Improvement of the sunflower bound for 1-intersecting constant dimension subspace codes

Leo Storme

Ghent University
Dept. of Mathematics
Krijgslaan 281
9000 Ghent
Belgium
(joint work with D. Bartoli, A. Riet and P. Vandendriessche)

Istanbul, November 6, 2015

t-INTERSECTING CONSTANT DIMENSION SUBSPACE CODES

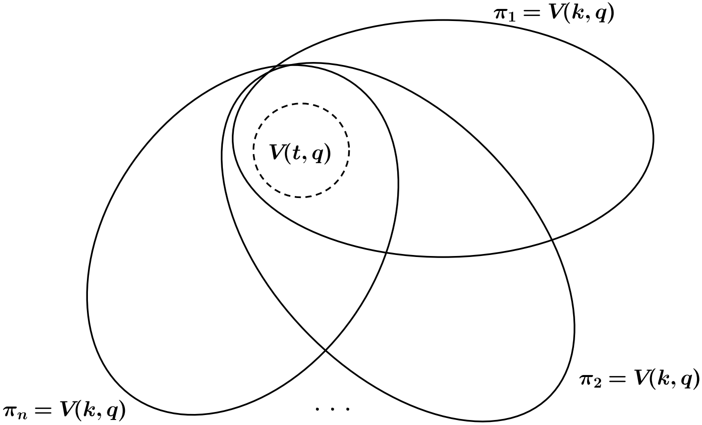
t-Intersecting constant dimension subspace code:

- Codewords are k-dimensional vector spaces.
- Distinct codewords intersect in t-dimensional vector spaces.

Classical example:

- **Sunflower:** all codewords pass through same t-dimensional vector space.

SUNFLOWER



LARGE t-INTERSECTING CONSTANT DIMENSION SUBSPACE CODES

THEOREM

Large t-intersecting constant dimension subspace codes are sunflowers.
If

$$|C| > \left(\frac{q^k - q^t}{q-1}\right)^2 + \left(\frac{q^k - q^t}{q-1}\right) + 1,$$

then C is sunflower.

Conjecture:
Let C be t-intersecting constant dimension subspace code.
If

$$|C| > q^k + q^{k-1} + \cdots + q + 1,$$

then C is sunflower.

Code C of 1-intersecting 3-dimensional spaces in $V(6,2)$.

- **Conjecture:** If $|C| > 15$, then C is sunflower.
- **Counterexample 1: (Etzion and Raviv)**
  Code C of size 16 which is not sunflower.
- **Counterexample 2: (Bartoli and Pavese)**
  Code C of 1-intersecting 3-dimensional spaces in $V(6,2)$ has size at most 20, and unique example of size 20.

Improvement to upper bound for $t = 1$

- If
$$|C| > \left(\frac{q^k - q^t}{q-1}\right)^2 + \left(\frac{q^k - q^t}{q-1}\right) + 1,$$
then C is sunflower.
- For $t = 1$, if
$$|C| > \left(\frac{q^k - q}{q-1}\right)^2 + \left(\frac{q^k - q}{q-1}\right) + 1,$$
then C is sunflower.
- **Question:** Can this bound be improved?

Improvement to upper bound for $t = 1$

Assumptions:

- C = 1-intersecting constant dimension code of k-spaces.
- C not sunflower.
- $$|C| = \left(\frac{q^k - q}{q-1}\right)^2 + \left(\frac{q^k - q}{q-1}\right) + 1 - \delta,$$
  with
  $$\delta = q^{k-2}.$$

Improvement to upper bound for $t = 1$

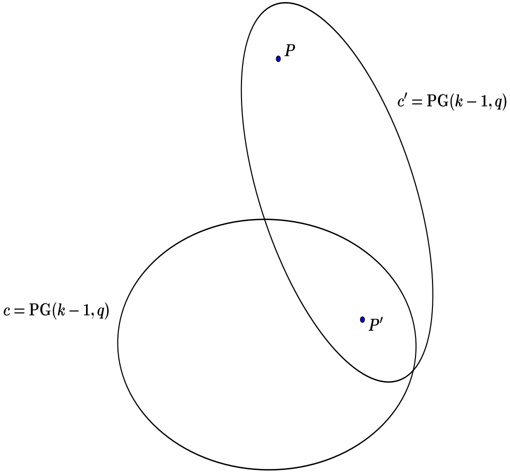
See codeword $c \in C$ as $\mathrm{PG}(k-1, q)$.
Define

$$\mathcal{S} = \cup_{c \in C} c.$$

Lemma

Point $P \in \mathcal{S}$ belongs to at most $\frac{q^k - 1}{q - 1}$ codewords.

Improvement to upper bound for $t = 1$



Improvement to upper bound for $t = 1$

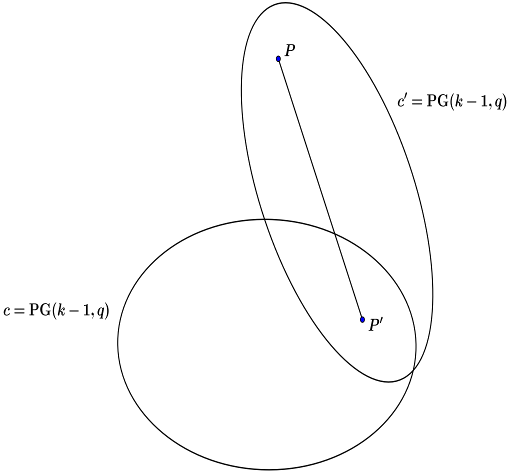
Lemma

If $|C| > (\frac{q^k-q}{q-1})^2$, then every codeword in C has at least one point in $\frac{q^k-1}{q-1}$ codewords.

Lemma

If point P lies in $\frac{q^k-1}{q-1}$ codewords, then line through P and other point of $\mathcal{S}$ is completely contained in $\mathcal{S}$.

Improvement to upper bound for $t = 1$



Improvement to upper bound for $t = 1$

Lemma

If point P lies in $\frac{q^k-1}{q-1}$ codewords, then

$$|\mathcal{S}| = |\cup_{c \in C}\, c| = \left(\frac{q^k - q}{q-1}\right)^2 + \left(\frac{q^k - q}{q-1}\right) + 1.$$

REMARK:

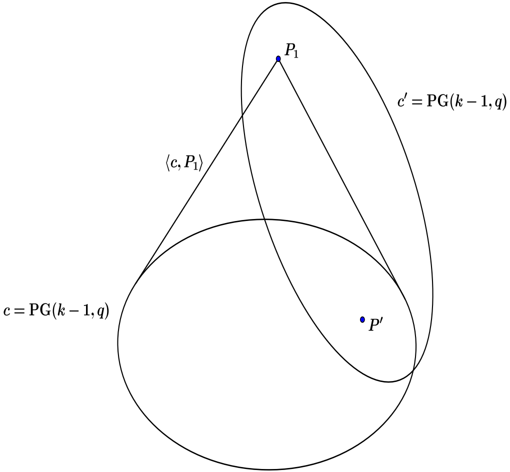
$$\left(\frac{q^k - q}{q-1}\right)^2 + \left(\frac{q^k - q}{q-1}\right) + 1 \neq |\mathrm{PG}(T, q)|.$$

$$|\mathrm{PG}(2k-2, q)| < |\mathcal{S}| < |\mathrm{PG}(2k-1, q)|.$$

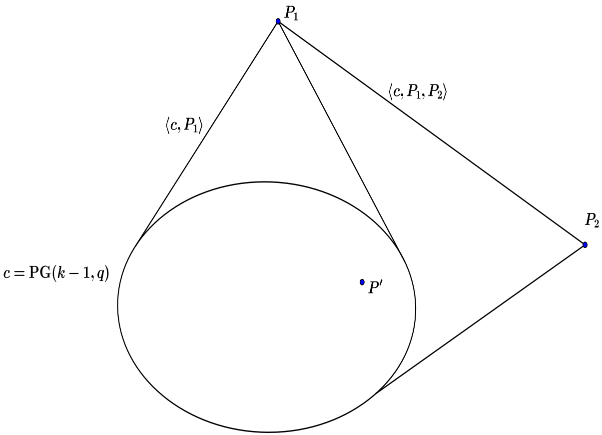
LEMMA

If more than $q^{2k-3} + q^{2k-4} + \cdots + q + 1$ points of $\mathcal{S}$ lie in $\frac{q^k-1}{q-1}$ codewords, then $(2k-2)$-dimensional subspace contained in $\mathcal{S}$.

Improvement to upper bound for $t = 1$



Improvement to upper bound for $t = 1$



Eventually

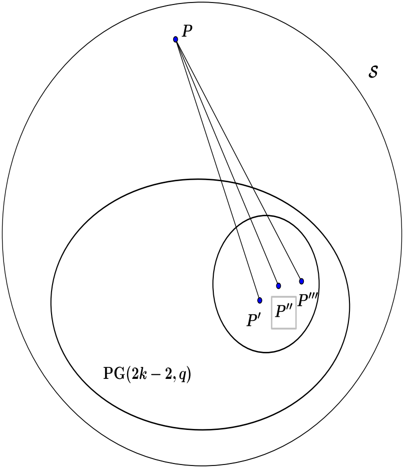
$$\mathrm{PG}(2k-2,q) = \langle c, P_1, \ldots, P_{k-1} \rangle \subset \mathcal{S}.$$

But

$$|\mathcal{S}| > |\mathrm{PG}(2k-2,q)|.$$

$$|\mathcal{S} \setminus \mathrm{PG}(2k-2,q)| \approx q^{2k-3}.$$

- If point of $\mathcal{S} \setminus \mathrm{PG}(2k-2, q)$ in $\frac{q^k-1}{q-1}$ codewords, then $\mathrm{PG}(2k-1, q) \subset \mathcal{S}$. (FALSE)
- So all points of $\mathcal{S} \setminus \mathrm{PG}(2k-2, q)$ in less than $\frac{q^k-1}{q-1}$ codewords.



P
$\mathcal{S}$
P'
P''
P'''
$\mathrm{PG}(2k-2, q)$

Improvement to upper bound for $t = 1$

- All points of $\mathcal{S} \setminus \mathrm{PG}(2k-2, q)$ in less than $\frac{q^k-1}{q-1}$ codewords.
-
$$|\mathcal{S} \setminus \mathrm{PG}(2k-2, q)| \approx q^{2k-3}.$$
- So number of points of $\mathcal{S}$ in $\frac{q^k-1}{q-1}$ codewords, is approximately q^{2k-4}. (TOO SMALL)

Improvement to upper bound for $t = 1$

Theorem (Bartoli, Riet, Storme, Vandendriessche)

Every 1-intersecting constant dimension code C of codewords of dimension k of size

$$|C| = \left(\frac{q^k - q}{q-1}\right)^2 + \left(\frac{q^k - q}{q-1}\right) + 1 - \delta,$$

with

$$\delta = q^{k-2},$$

is sunflower.

Thank you very much for your attention!

UNIVERSITEIT GENT