Construction of new large sets of designs over the binary field

Alfred Wassermann

Department of Mathematics, Universität Bayreuth, Germany

joint work with Michael Kiermaier and Reinhard Laue

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Outline

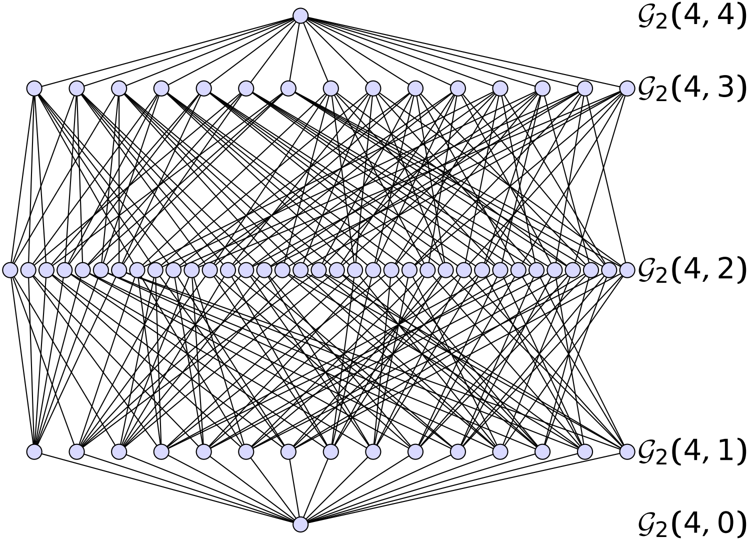
- Designs over finite fields
- Computer construction
- Infinite series of large sets

Designs over finite fields

Subspaces

- vector space $\mathcal{V} = \mathbb{F}_q^v$
- Grassmannian: $\mathcal{G}_q(v, k) := \{U \le \mathbb{F}_q^v : \dim U = k\}$

Subspace lattice of $\mathbb{F}_2^4$



Subspace lattice

- Gaussian coefficient:

$$\begin{bmatrix} v \\ k \end{bmatrix}_q = \frac{(q^v - 1)(q^{v-1} - 1)\cdots(q^{v-k+1} - 1)}{(q^k - 1)(q^{k-1} - 1)\cdots(q - 1)}$$

- $|\mathcal{G}_q(v,k)| = \begin{bmatrix} v \\ k \end{bmatrix}_q$

Designs over finite fields

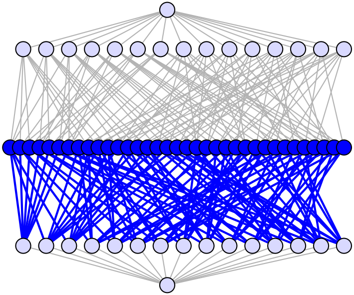
- Cameron (1974), Delsarte (1976)
- $\mathcal{B} \subseteq \mathcal{G}_q(v, k)$: set of k-subspaces (blocks)
- $(\mathbb{F}_q^v, \mathcal{B})$: t-$(v, k, \lambda; q)$ design over $\mathbb{F}_q$

  each t-subspace of $\mathbb{F}_q^v$ is contained in exactly λ blocks of $\mathcal{B}$

- $\mathcal{B}$ set: simple design
- $\mathcal{B}$ multiset: non-simple design

Designs over finite fields

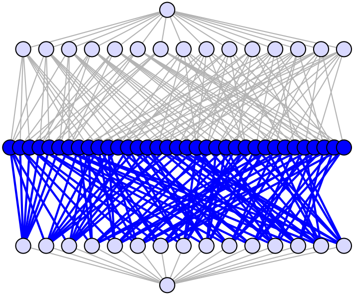
- $\mathcal{B} = \mathcal{G}_q(v,k)$ is a t-$(v,k,\begin{bmatrix} v-t \\ k-t \end{bmatrix}_q;q)$ design: trivial design



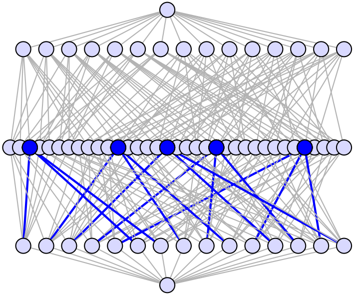
trivial 1-$(4,2,7;2)$ design

Designs over finite fields

- $\mathcal{B} = \mathcal{G}_q(v,k)$ is a t-$(v,k,\left[\begin{smallmatrix} v-t \\ k-t \end{smallmatrix}\right]_q;q)$ design: trivial design



trivial 1-$(4,2,7;2)$ design



1-$(4,2,1;2)$ design

t-$(v, k, \lambda; q)$ designs

- $|\mathcal{B}| = \lambda \frac{\begin{bmatrix} v \\ t \end{bmatrix}_q}{\begin{bmatrix} k \\ t \end{bmatrix}_q}$
- Necessary conditions:

$$\lambda_i = \lambda \frac{\begin{bmatrix} v-i \\ t-i \end{bmatrix}_q}{\begin{bmatrix} k-i \\ t-i \end{bmatrix}_q} \in \mathbb{Z} \qquad \text{for } i = 0, \ldots, t$$

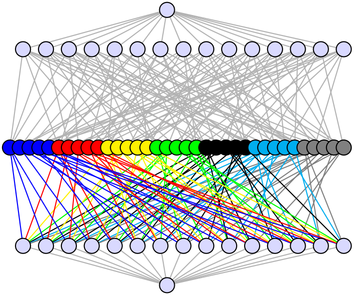
Related designs

t-$(v, k, \lambda; q)$ design $\rightarrow$

- dual design: t-$(v, v-k, \lambda; q)$
- derived design: $(t-1)$-$(v-1, k-1, \lambda; q)$
- residual design: $(t-1)$-$(v-1, k, \mu; q)$, where $\mu = \lambda \cdot \begin{bmatrix} v-k \\ 1 \end{bmatrix}_q / \begin{bmatrix} k-t+1 \\ 1 \end{bmatrix}_q$

Large sets of q-analogs of designs

- $\mathcal{G}_q(v,k)$ is a t-$(v,k,\begin{bmatrix}v-t\\k-t\end{bmatrix}_q;q)$ design
- Large set $LS_q[N](t,k,v)$:
  partition of $\mathcal{G}_q(v,k)$ into N disjoint t-$(v,k,\lambda;q)$ designs



$LS_2[7](1,2,4)$

- Necessary: $N \cdot \lambda = \begin{bmatrix}v-t\\k-t\end{bmatrix}_q$

Automorphisms

Designs over finite fields:

- $GL(v, q) = \{M \in \mathbb{F}_q^{v \times v} : M \text{ invertible}\}$
- $\sigma \in GL(v, q)$ automorphism: $\mathcal{B}^\sigma = \mathcal{B}$

Automorphisms of designs over finite fields

- Singer cycle:
  - take $v \in \mathbb{F}_q^v$ as an element of $\mathbb{F}_{q^v}$
  - $(\mathbb{F}_{q^v} \setminus \{0\}, \cdot)$ is a cyclic group G of order $q^v - 1$, i.e.
  - $G = \langle \sigma \rangle$
  - $G \leq GL(v, q)$ is called *Singer cycle*
- Frobenius automorphism:
  - $\phi : \mathbb{F}_{q^v} \to \mathbb{F}_{q^v}$, $U \mapsto U^q$
  - $|\langle \phi \rangle| = v$
- $|\langle \sigma, \phi \rangle| = v \cdot (q^v - 1)$

Computer construction

Brute force approach for construction

- incidence matrix between t-subset and k-subsets:

$$M_{t,k} = (m_{i,j}), \text{ where } m_{i,j} = \begin{cases} 1 & \text{if } T_i \subset K_j \\ 0 & \text{else} \end{cases}$$

- solve

$$M_{t,k} \cdot x = \begin{pmatrix} \lambda \\ \lambda \\ \vdots \\ \lambda \end{pmatrix} \quad \text{for 0/1-vector } x$$

Designs with prescribed automorphism group

Construction of designs with prescribed automorphism group

- choose group G acting on $\mathcal{V}$, i.e. $G \leq S_v$
- search for t-designs $\mathcal{D} = (\mathcal{V}, \mathcal{B})$ having G as a group of automorphisms,
  i.e. for all

$$g \in G \text{ and } K \in \mathcal{B} \Longrightarrow K^g \in \mathcal{B}.$$

- construct $\mathcal{D} = (\mathcal{V}, \mathcal{B})$ as

union of orbits of G on k-subsets.

The method of Kramer and Mesner

Definition

- $K \subset \mathcal{V}$ and $|K| = k$: $K^G := \{K^g \mid g \in G\}$
- $T \subset \mathcal{V}$ and $|T| = t$: $T^G := \{T^g \mid g \in G\}$
- Let
$$K_1^G \cup K_2^G \cup \ldots \cup K_n^G \subseteq \binom{\mathcal{V}}{k}$$
and
$$T_1^G \cup T_2^G \cup \ldots \cup T_m^G = \binom{\mathcal{V}}{t}$$
-
$$M_{t,k}^G = (m_{i,j}) \text{ where } m_{i,j} := |\{K \in K_j^G \mid T_i \subset K\}|$$

The method of Kramer and Mesner

Theorem (Kramer and Mesner, 1976)

The union of orbits corresponding to the 1*s in a* $\{0,1\}$ *vector which solves*

$$M_{t,k}^{G} \cdot x = \begin{pmatrix} \lambda \\ \lambda \\ \vdots \\ \lambda \end{pmatrix}$$

is a t*-*(v,k,λ) *design having* G *as an automorphism group.*

Known large sets for $t \geq 2$

- $LS_2[3](2, 3, 8)$: Braun, Kohnert, Östergård, W. (2014)
  - Three disjoint 2-$(8, 3, 21; 2)$ designs
  - Group: $\langle\sigma\rangle$ in $GL(8, 2)$ of order 255
- $LS_3[2](2, 3, 6)$: Braun (2005)
  - Two disjoint 2-$(6, 3, 20; 3)$ designs
- $LS_5[2](2, 3, 6)$: Braun, Kiermaier, Kohnert, Laue (2014)
  - Two disjoint 2-$(6, 3, 78; 5)$ designs

A new large set

- $LS_2[3](2,4,8)$
  - Three disjoint 2-$(8,4,217;2)$ designs
  - Group: $\langle \sigma^5, \phi^2 \rangle$ in $GL(8,2)$ of order 204

Related large sets

Theorem (Kiermaier, Laue 2015)

- *derived large set:*

$$LS_q[N](t,k,v) \quad \rightarrow \quad LS_q[N](t-1,k-1,v-1)$$

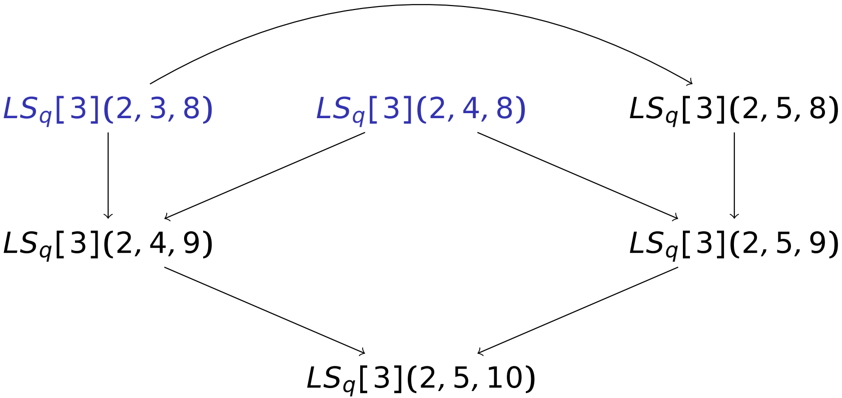
- *q-analog of Van Trung, Van Leyenhorst, Driessen:*

$$LS_q[N](t,k-1,v-1) \quad \text{and} \quad LS_q[N](t,k,v-1)$$

$$\longrightarrow$$

$$LS_q[N](t,k,v)$$

Related large sets



Admissibility and realizability of $LS_2[3](2, k, v)$

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-																		7
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-	-	5																10
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-	-	5	?	?	?	?	?	11										22
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3	4	5	-	-	-	9	10	11	?	?								26
-	4	5	-	-	-	-	10	11	?	?								27
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-	4	5	?	?	?	?	10	11	?	?	?	?	16	17	-	-		39
-	-	5	?	?	?	?	?	11	?	?	?	?	?	17	-	-	-	40

Open problems

Thank you for listening !