Designs on which the unitary group $U(3,3)$ acts transitively

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A t-(v, k, λ) **design** is a finite incidence structure $\mathcal{D} = (\mathcal{P}, \mathcal{B}, \mathcal{I})$ satisfying the following requirements:

1. $|\mathcal{P}| = v$,
2. every element of $\mathcal{B}$ is incident with exactly k elements of $\mathcal{P}$,
3. every t elements of $\mathcal{P}$ are incident with exactly λ elements of $\mathcal{B}$.

Every element of $\mathcal{P}$ is incident with exactly $r = \frac{\lambda(v-1)}{k-1}$ elements of $\mathcal{B}$.

The number of blocks is denoted by b.

If $b = v$ (or equivalently $k = r$) then the design is called **symmetric**.

- A 2-(v, k, λ) design is called a block design.
- If $\mathcal{D}$ is a t-design, then it is also a s-design, for $1 \leq s \leq t - 1$.
- An **incidence matrix** of a design $\mathcal{D}$ is a matrix $A = [a_{ij}]$ where $a_{ij} = 1$ if jth point is incident with the ith block and $a_{ij} = 0$ otherwise.

J. D. Key, J. Moori:

- Construction method of primitive symmetric designs (and regular graphs) for which a stabilizer of a point and a stabilizer of a block are conjugate.

Theorem 1 [D. Crnković, V. Mikulić Crnković]

Let G be a finite permutation group **acting primitively on the sets Ω_1 and Ω_2 of size m and n, respectively**. Let $\alpha \in \Omega_1$ and $\Delta_2 = \bigcup_{i=1}^{s} \delta_i G_\alpha$, where $\delta_1, ..., \delta_s \in \Omega_2$ are representatives of distinct G_α-orbits. If $\Delta_2 \neq \Omega_2$ and

$$\mathcal{B} = \{\Delta_2 g : g \in G\},$$

then $\mathcal{D}(G, \alpha, \delta_1, ..., \delta_s) = (\Omega_2, \mathcal{B})$ is a **design** 1-$(n, |\Delta_2|, \sum_{i=1}^{s} |\alpha G_{\delta_i}|)$ with m blocks, and G acts as an **automorphism group, primitively on points and blocks** of the design.

This construction gives us **all 1-designs on which the group G acts primitively on points and blocks**.

Corollary 1

If a group G acts primitively on the points and the blocks of a 1-design $\mathcal{D}$, then $\mathcal{D}$ can be obtained as described in Theorem 1, *i.e.*, such that Δ_2 is a union of G_α-orbits.

Theorem 2 [D. Crnković, V. Mikulić Crnković, AŠ]

Let G be a finite permutation group **acting transitively** on the sets Ω_1 and Ω_2 of size m and n, respectively. Let $\alpha \in \Omega_1$ and $\Delta_2 = \bigcup_{i=1}^{s} \delta_i G_\alpha$, where $\delta_1, ..., \delta_s \in \Omega_2$ are representatives of distinct G_α-orbits. If $\Delta_2 \neq \Omega_2$ and

$$\mathcal{B} = \{\Delta_2 g : g \in G\},$$

then the incidence structure $\mathcal{D}(G, \alpha, \delta_1, ..., \delta_s) = (\Omega_2, \mathcal{B})$ is a 1-$(n, |\Delta_2|, \frac{|G_\alpha|}{|G_{\Delta_2}|} \sum_{i=1}^{s} |\alpha G_{\delta_i}|)$ design with $\frac{m \cdot |G_\alpha|}{|G_{\Delta_2}|}$ blocks. Then the group $H \cong G / \bigcap_{x \in \Omega_2} G_x$ acts as an automorphism group on $(\Omega_2, \mathcal{B})$, **transitively on points and blocks** of the design.

Corollary 2

If a group G acts transitively on the points and the blocks of a 1-design $\mathcal{D}$, then $\mathcal{D}$ can be obtained as described in the Theorem 2, i.e., such that Δ_2 is a union of G_α-orbits.

Using the described approach a number of 2-designs and strongly regular graphs from the groups $U(3,3)$, $U(3,4)$, U(3,5), $U(3,7)$, $U(4,2)$, $U(4,3)$, $U(5,2)$, $L(2,32)$, $L(2,49)$, $L(3,5)$, $L(4,3)$ and $S(6,2)$ have been constructed.

Table: Properties of the subgroups of the group $U(3,3)$

Structure	Order	Index	Size of the class
I	1	6048	1
Z_2	2	3024	63
Z_3	3	2016	28
Z_3	3	2016	336
Z_7	7	864	288
Z_4	4	1512	63
$Z_2 \times Z_2$	4	1512	63
Z_4	4	1512	189
Z_6	6	1008	252
S_3	6	1008	336
$Z_3 \times Z_3$	9	672	112
$Z_7 : Z_3$	21	288	288
Q_8	8	756	63
D_8	8	756	189
$Z_4 \times Z_2$	8	756	189
Z_8	8	756	378
Z_{12}	12	504	252
A_4	12	504	252

Table: Properties of the subgroups of the group $U(3,3)$

Structure	Size	Index	Size of the class
$Z_3 \times S_3$	18	336	336
$(Z_3 \times Z_3) : Z_3$	27	224	28
$(Z_4 \times Z_2) : Z_2$	16	378	63
$Z_4 \times Z_4$	16	378	63
$Z_8 : Z_2$	16	378	189
$SL(2,3)$	24	252	63
S_4	24	252	252
$Z_3 : Z_8$	24	252	252
$((Z_3 \times Z_3) : Z_3) : Z_2$	54	112	28
$(Z_4 \times Z_4) : Z_2$	32	189	189
$(Z_4 \times Z_4) : Z_3$	48	126	63
$SL(2,3) : Z_2$	48	126	63
$((Z_3 \times Z_3) : Z_3) : Z_4$	108	56	28
$PSL(3,2)$	168	36	36
$((Z_4 \times Z_4) : Z_3) : Z_2$	96	63	63
$SL(2,3) : Z_4$	96	63	63
$((Z_3 \times Z_3) : Z_3) : Z_8$	216	28	28
$PSU(3,3)$	6048	1	1

In order to obtain **block designs** we have to make some basic steps:

- Determine the set of points.
- Make a list of all possible base blocks.
- Solve the problem of huge number of constructed designs with the same parameters (isomorphic or not?).

We had to add some extra eliminations.

Table: Block designs constructed from the group $U(3,3)$

Parameters of block designs	# non-isomorphic	Full automorphism group
2-(28, 3, 2)	1	$U(3,3):Z_2$
2-(28, 3, 8)	1	$U(3,3):Z_2$
2-(28, 3, 16)	1	$S(6,2)$
2-(28, 4, 1)	1	$U(3,3):Z_2$
2-(28, 4, 4)	1	$U(3,3):Z_2$
2-(28, 4, 32)	1	$U(3,3):Z_2$
2-(28, 4, 48)	2	$U(3,3):Z_2$
2-(28, 4, 96)	2	$U(3,3):Z_2$
2-(28, 5, 40)	1	$U(3,3):Z_2$
2-(28, 5, 80)	4	$U(3,3):Z_2$
	1	$U(3,3)$
2-(28, 5, 160)	2	$U(3,3)$
	8	$U(3,3):Z_2$
	1	$S(6,2)$
2-(28, 6, 20)	1	$U(3,3):Z_2$
2-(28, 6, 30)	1	$U(3,3):Z_2$
2-(28, 6, 40)	2	$U(3,3):Z_2$
	1	$S(6,2)$
2-(28, 6, 60)	3	$U(3,3):Z_2$
2-(28, 6, 80)	1	$U(3,3)$
	1	$U(3,3):Z_2$
	1	$S(6,2)$
2-(28, 6, 120)	2	$U(3,3)$
	3	$U(3,3):Z_2$
2-(28, 6, 240)	20	$U(3,3)$
	16	$U(3,3):Z_2$

Table: Block designs constructed from the group $U(3,3)$

Parameters of block designs	# non-isomorphic	Full automorphism group
2-(28, 7, 16)	1	$S(6,2)$
2-(28, 7, 48)	1	$U(3,3):Z_2$
2-(28, 7, 56)	3	$U(3,3):Z_2$
2-(28, 7, 84)	1	$U(3,3):Z_2$
2-(28, 7, 112)	2	$U(3,3):Z_2$
	1	$U(3,3)$
2-(28, 7, 168)	5	$U(3,3):Z_2$
	8	$U(3,3)$
2-(28, 7, 336)	37	$U(3,3):Z_2$
	73	$U(3,3)$
2-(28, 8, 14)	1	$U(3,3):Z_2$
2-(28, 8, 56)	3	$U(3,3):Z_2$
2-(28, 8, 112)	2	$U(3,3):Z_2$
2-(28, 8, 224)	12	$U(3,3):Z_2$
	11	$U(3,3)$
2-(28, 8, 448)	217	$U(3,3)$
	61	$U(3,3):Z_2$
	1	$S(6,2)$
2-(28, 9, 32)	1	$U(3,3):Z_2$
2-(28, 9, 72)	1	$U(3,3):Z_2$
	1	$U(3,3)$
2-(28, 9, 96)	1	$U(3,3):Z_2$
	1	$U(3,3)$
2-(28, 9, 144)	1	$U(3,3):Z_2$

Table: Block designs constructed from the group $U(3,3)$

Parameters of block designs	# non-isomorphic	Full automorphism group
2-(28, 9, 192)	5	$U(3,3) : Z_2$
	4	$U(3,3)$
2-(28, 9, 288)	11	$U(3,3) : Z_2$
	22	$U(3,3)$
2-(28, 9, 576)	103	$U(3,3) : Z_2$
	503	$U(3,3)$
2-(28, 10, 40)	1	$S(6,2)$
2-(28, 10, 45)	1	$S(6,2)$
2-(28, 10, 60)	1	$U(3,3) : Z_2$
2-(28, 10, 90)	3	$U(3,3) : Z_2$
2-(28, 10, 120)	1	$U(3,3) : Z_2$
	1	$U(3,3)$
2-(28, 10, 180)	3	$U(3,3) : Z_2$
	3	$U(3,3)$
2-(28, 10, 240)	4	$U(3,3) : Z_2$
	4	$U(3,3)$
2-(28, 10, 360)	21	$U(3,3) : Z_2$
	24	$U(3,3)$
2-(28, 10, 720)	136	$U(3,3) : Z_2$
	996	$U(3,3)$

Table: Block designs constructed from the group $U(3,3)$

Parameters of block designs	# non-isomorphic	Full automorphism group
2-(28, 11, 110)	1	$S(6,2)$
	1	$U(3,3)$
2-(28, 11, 220)	1	$U(3,3):Z_2$
2-(28, 11, 440)	18	$U(3,3):Z_2$
	44	$U(3,3)$
2-(28, 11, 880)	1650	$U(3,3)$
	195	$U(3,3):Z_2$
	2	$S(6,2)$
2-(28, 12, 11)	1	$S(6,2)$
2-(28, 12, 44)	1	$U(3,3):Z_2$
2-(28, 12, 88)	1	$U(3,3):Z_2$
2-(28, 12, 132)	4	$U(3,3):Z_2$
2-(28, 12, 176)	1	$U(3,3):Z_2$
	1	$U(3,3)$
2-(28, 12, 264)	3	$U(3,3):Z_2$
	1	$U(3,3)$
2-(28, 12, 352)	8	$U(3,3):Z_2$
	6	$U(3,3)$
2-(28, 12, 528)	24	$U(3,3):Z_2$
	46	$U(3,3)$
2-(28, 12, 1056)	218	$U(3,3):Z_2$
	2372	$U(3,3)$
	1	$S(6,2)$

Table: Block designs constructed from the group $U(3,3)$

Parameters of block designs	# non-isomorphic	Full automorphism group
2-(28, 13, 104)	1	$U(3,3):Z_2$
2-(28, 13, 208)	2	$U(3,3):Z_2$
	1	$U(3,3)$
	1	$S(6,2)$
2-(28, 13, 312)	1	$U(3,3):Z_2$
	1	$U(3,3)$
2-(28, 13, 416)	7	$U(3,3):Z_2$
	6	$U(3,3)$
	1	$S(6,2)$
2-(28, 13, 624)	19	$U(3,3):Z_2$
	59	$U(3,3)$
2-(28, 13, 1248)	260	$U(3,3):Z_2$
	2887	$U(3,3)$
2-(28, 14, 182)	1	$U(3,3)$
2-(28, 14, 208)	2	$U(3,3):Z_2$
2-(28, 14, 364)	14	$U(3,3):Z_2$
2-(28, 14, 728)	28	$U(3,3):Z_2$
	53	$U(3,3)$
2-(28, 14, 1456)	246	$U(3,3):Z_2$
	3016	$U(3,3)$

Table: 3-designs constructed from the group $U(3,3)$

Parameters of designs	# non-isomorphic	Full automorphism group
3-$(28, 13, 528)$	40	$U(3,3)$
3-$(28, 14, 84)$	1	$U(3,3)$
3-$(28, 14, 168)$	2	$U(3,3) : Z_2$
3-$(28, 14, 336)$	7	$U(3,3)$
3-$(28, 14, 672)$	12	$U(3,3) : Z_2$
	136	$U(3,3)$

A t-(v, k, λ) design $\mathcal{D}$ is quasi-symmetric with intersection numbers x and y $(x < y)$, if any two blocks of $\mathcal{D}$ intersect in either x or y points.

Table: Quasi-symmetric designs constructed from the group $U(3, 3)$

Parameters of designs	Full automorphism group
2-$(28, 4, 1)$	$U(3, 3) : Z_2$
2-$(28, 12, 11)$	$S(6, 2)$

Thank you for your attention!