A geometrical bound for the sunflower property

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DARNEC 2015, November 4, 2015
1. \textit{t-Intersecting constant dimension random network codes}

- A dimension result
$t$-Intersecting constant dimension random network codes

A dimension result

1 $t$-INTERSECTING CONSTANT DIMENSION RANDOM NETWORK CODES

A dimension result
**t-Intersecting constant dimension random network codes**

**t-Intersecting constant dimension random network code:**
- Codewords are $k$-dimensional vector spaces.
- Distinct codewords intersect in $t$-dimensional vector spaces.

**Classical example:**
- **Sunflower:** all codewords pass through same $t$-dimensional vector space.
SUNFLOWER

\[ \pi_1 = V(k, q) \]

\[ V(t, q) \]

\[ \pi_n = V(k, q) \]

\[ \pi_2 = V(k, q) \]

\[ \ldots \]
Large $t$-intersecting constant dimension random network codes are sunflowers.

**Proof:** Via result from classical coding theory.
$t$-intersecting binary constant weight codes

- Binary constant weight code: codewords have fixed weight $w$.
- $t$-intersecting binary constant weight code: $|\text{supp}(c_1 \cap c_2)| = t$.
- **Sunflower**: all codewords have ones in $t$ fixed positions.
\( t \)-intersecting constant weight code \( C \).

**Theorem**

If \(|C| > (w - t)^2 + (w - t) + 1\), then \( C \) is sunflower.

**Corollary**

If \( t = 1 \) and \(|C| = (w - t)^2 + (w - t) + 1\), then \( C \) is sunflower or set of incidence vectors of projective plane of order \( w \).
Let \( c_1 \in C \), then \( c_1 = V(k, q) \equiv PG(k - 1, q) \).

Identify \( c_1 \) with its binary incidence vector of weight \( \frac{q^k - 1}{q - 1} \).

Then \( |\text{supp}(c_1 \cap c_2)| = |PG(t - 1, q)| = \frac{q^t - 1}{q - 1} \).

So \( C \) is transformed into binary \( \left( \frac{q^t - 1}{q - 1} \right) \)-intersecting binary constant weight code with \( w = \frac{q^k - 1}{q - 1} \).

So, if \( |C| > \left( \frac{q^k - q^t}{q - 1} \right)^2 + \left( \frac{q^k - q^t}{q - 1} \right) + 1 \), then \( C \) is sunflower.
Improvement to upper bound for $t = 1$

(Bartoli, Riet, Storme, Vandendriessche)

Assumptions:

- $C$ = 1-intersecting constant dimension code of $k$-spaces.
- $C$ not sunflower.

$$|C| \leq \left(\frac{q^k - q}{q - 1}\right)^2 + \left(\frac{q^k - q}{q - 1}\right) + 1 - q^{k-2}.$$
**Conjecture**

Let $C$ be $t$-intersecting constant dimension random network code.
If

$$|C| > q^k + q^{k-1} + \cdots + q + 1,$$

then $C$ is sunflower.
**Counterexamples to Conjecture**

Code $C$ of 1-intersecting 3-dimensional spaces in $V(6, 2)$.

- **Conjecture:** If $|C| > 15$, then $C$ is sunflower.
- **Counterexample 1: (Etzion and Raviv)**
  Code $C$ of size 16 which is not sunflower.
- **Counterexample 1: (Bartoli and Pavese)**
  Code $C$ of 1-intersecting 3-dimensional spaces in $V(6, 2)$ has size at most 20, and unique example of size 20.
Let $C$ be $(k - t)$-intersecting constant dimension random network code of $k$-dimensional codewords.

Let $C = \{\pi_1, \ldots, \pi_n\}$.

Maximal dimension for sunflower is

$$\dim \langle \pi_1, \ldots, \pi_n \rangle = k + t(n - 1).$$

**Question:** From which dimension for $\langle \pi_1, \ldots, \pi_n \rangle$ are we sure that $C$ is sunflower?
THEOREM (Barrolleta, De Boeck, Storme, Suárez-Canedo, Vandendriessche)

If $\dim\langle \pi_1, \ldots, \pi_n \rangle \geq k + (t - 1)(n - 1) + 2$, then $C$ is sunflower.

PROOF:

- Order codewords.

\[ \delta_i = \dim\langle \pi_1, \ldots, \pi_i \rangle - \dim\langle \pi_1, \ldots, \pi_{i-1} \rangle. \]

- Order codewords so that $\delta_2 \geq \delta_3 \geq \cdots \geq \delta_n$.

- Sequence $(\delta_2, \ldots, \delta_n)$.

- $\delta_2, \ldots, \delta_{n-1} \geq t - 1$. 

Leo Storme | Random network coding
**Theorem (Barrolleta, De Boeck, Storme, Suárez-Canedo, Vandendriessche)**

If \( \dim \langle \pi_1, \ldots, \pi_n \rangle \geq k + (t - 1)(n - 1) + 2 \), then \( C \) is sunflower.

**Theorem (Barrolleta, De Boeck, Storme, Suárez-Canedo, Vandendriessche)**

If \( \dim \langle \pi_1, \ldots, \pi_n \rangle = k + (t - 1)(n - 1) + 1 \), then \( C \) is sunflower, or one of two other types of examples.
**DIMENSION RESULT IS SHARP**

\[ W_1 = [k - t + 1], \quad V = [k - t + 2] \]

\[ W_2 = [k - t + 1] \]

\[ \vdots \]

\[ W_n = [k - t + 1] \]

\[ X_1 = [t - 1] \]

\[ X_2 = [t - 1] \]

\[ \vdots \]

\[ X_n = [t - 1] \]
A dimension result

**Dimension result is sharp**

- \( V = [k - t + 2] \) fixed.
- \( W_1, \ldots, W_n \) are \([k - t + 1]\) in \( V \), not through common \([k - t]\).
- \( X_1, \ldots, X_n \) are \([t - 1]\), and
- codewords are \( \pi_i = \langle W_i, X_i \rangle, i = 1, \ldots, n. \)

\((\delta_2, \ldots, \delta_n) = (t, t - 1, \ldots, t - 1).\)
**Dimension result is sharp**

- $V = [k - t]$
- $W = [k - t - 1]$
- $X = [t + 1 - m]$
- $M_{n-1} = [t - 1]$
- $M_{m+1} = [t - 1]$
- $N_1 = [t]$
- $n_1$
- $N_m = [t]$
- $n_m$
- $p_{m+1}$
- $p_{n-1}$
- $\langle N_1, \ldots, N_m \rangle$
**Dimension result is sharp**

- **Type 1:** \( \pi_1 = \langle V, N_1 \rangle, \ldots, \pi_m = \langle V, N_m \rangle. \)
- **Type 2:** 
  \[ \pi_{m+1} = \langle V, M_{m+1}, p_{m+1} \rangle, \ldots, \pi_{n-1} = \langle V, M_{n-1}, p_{n-1} \rangle. \]
- **Type 3:** \( \pi_n = \langle W, X, n_1, \ldots, n_m \rangle. \)

\[ (\delta_2, \ldots, \delta_n) = (t, \ldots, t, t - 1, \ldots, t - 1, t + 1 - m). \]
Thank you very much for your attention!