A geometrical bound for the sunflower property

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OUTLINE

1. t-INTERSECTING CONSTANT DIMENSION RANDOM NETWORK CODES
   - A dimension result

Outline

1. t-Intersecting constant dimension random network codes
   - A dimension result

t-Intersecting constant dimension random network codes

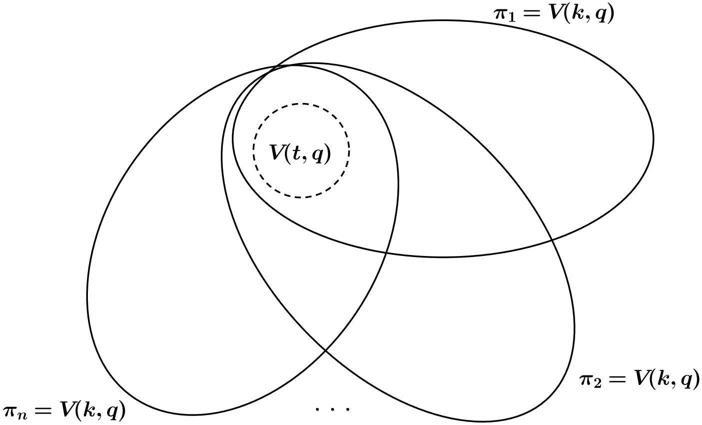
t-Intersecting constant dimension random network code:

- Codewords are k-dimensional vector spaces.
- Distinct codewords intersect in t-dimensional vector spaces.

Classical example:

- **Sunflower:** all codewords pass through same t-dimensional vector space.

SUNFLOWER



Large t-intersecting constant dimension random network codes

Theorem

Large t-intersecting constant dimension random network codes are sunflowers.

Proof: Via result from classical coding theory. □

t-Intersecting binary constant weight codes

t-intersecting binary constant weight codes

- Binary constant weight code: codewords have fixed weight w.
- t-intersecting binary constant weight code: $|\text{supp}(c_1 \cap c_2)| = t$.
- **Sunflower:** all codewords have ones in t fixed positions.

t-intersecting constant weight code C.

Theorem

If $|C| > (w-t)^2 + (w-t) + 1$, then C is sunflower.

Corollary

If $t = 1$ and $|C| = (w-t)^2 + (w-t) + 1$, then C is sunflower or set of incidence vectors of projective plane of order w.

Extension to t-intersecting constant dimension random network codes

- Let $c_1 \in C$, then $c_1 = V(k, q) \equiv \mathrm{PG}(k-1, q)$.
- Identify c_1 with its binary incidence vector of weight $\frac{q^k-1}{q-1}$.
- Then $|\mathrm{supp}(c_1 \cap c_2)| = |\mathrm{PG}(t-1, q)| = \frac{q^t-1}{q-1}$.
- So C is transformed into binary $(\frac{q^t-1}{q-1})$-intersecting binary constant weight code with $w = \frac{q^k-1}{q-1}$.
- So, if $|C| > (\frac{q^k-q^t}{q-1})^2 + (\frac{q^k-q^t}{q-1}) + 1$, then C is sunflower.

IMPROVEMENT TO UPPER BOUND FOR $t = 1$

(Bartoli, Riet, Storme, Vandendriessche)
Assumptions:

- C = 1-intersecting constant dimension code of k-spaces.
- C not sunflower.
- $$|C| \leq \left(\frac{q^k - q}{q - 1}\right)^2 + \left(\frac{q^k - q}{q - 1}\right) + 1 - q^{k-2}.$$

CONJECTURE

Conjecture:
Let C be t-intersecting constant dimension random network code.
If
$$|C| > q^k + q^{k-1} + \cdots + q + 1,$$
then C is sunflower.

Counterexamples to conjecture

Code C of 1-intersecting 3-dimensional spaces in $V(6,2)$.

- **Conjecture:** If $|C| > 15$, then C is sunflower.
- **Counterexample 1: (Etzion and Raviv)**
  Code C of size 16 which is not sunflower.
- **Counterexample 1: (Bartoli and Pavese)**
  Code C of 1-intersecting 3-dimensional spaces in $V(6,2)$ has size at most 20, and unique example of size 20.

A dimension result

- Let C be $(k-t)$-intersecting constant dimension random network code of k-dimensional codewords.
- Let $C = \{\pi_1, \ldots, \pi_n\}$.
- Maximal dimension for sunflower is

$$\dim\langle \pi_1, \ldots, \pi_n \rangle = k + t(n-1).$$

Question: From which dimension for $\langle \pi_1, \ldots, \pi_n \rangle$ are we sure that C is sunflower?

A dimension result

Theorem (Barrolleta, De Boeck, Storme, Suárez-Canedo, Vandendriessche)

If $\dim\langle \pi_1, \ldots, \pi_n\rangle \geq k + (t-1)(n-1) + 2$, *then* C *is sunflower.*

PROOF:

- Order codewords.
- $$\delta_i = \dim\langle \pi_1, \ldots, \pi_i\rangle - \dim\langle \pi_1, \ldots, \pi_{i-1}\rangle.$$
- Order codewords so that $\delta_2 \geq \delta_3 \geq \cdots \geq \delta_n$.
- Sequence $(\delta_2, \ldots, \delta_n)$.
- $\delta_2, \ldots, \delta_{n-1} \geq t - 1$. □

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Dimension result is sharp

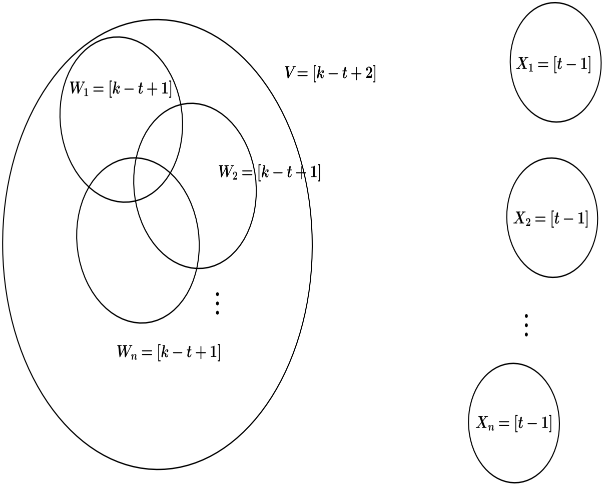
Theorem (Barrolleta, De Boeck, Storme, Suárez-Canedo, Vandendriessche)

If $\dim\langle \pi_1, \ldots, \pi_n \rangle \geq k + (t-1)(n-1) + 2$*, then* C *is sunflower.*

Theorem (Barrolleta, De Boeck, Storme, Suárez-Canedo, Vandendriessche)

If $\dim\langle \pi_1, \ldots, \pi_n \rangle = k + (t-1)(n-1) + 1$*, then* C *is sunflower, or one of two other types of examples.*

DIMENSION RESULT IS SHARP

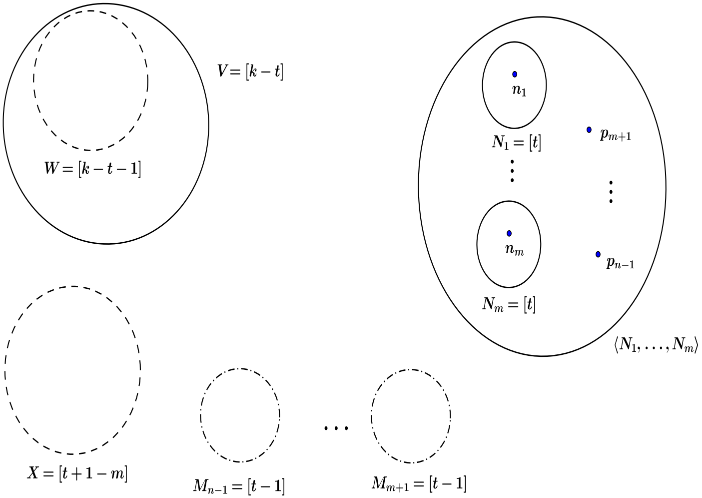


Dimension result is sharp

- $V = [k - t + 2]$ fixed.
- $W_1, \ldots, W_n$ are $[k - t + 1]$ in V, not through common $[k - t]$.
- $X_1, \ldots, X_n$ are $[t - 1]$, and
- codewords are $\pi_i = \langle W_i, X_i \rangle$, $i = 1, \ldots, n$.

$$(\delta_2, \ldots, \delta_n) = (t, t - 1, \ldots, t - 1).$$

Dimension result is sharp



DIMENSION RESULT IS SHARP

- Type 1: $\pi_1 = \langle V, N_1 \rangle, \ldots, \pi_m = \langle V, N_m \rangle$.
- Type 2:
  $\pi_{m+1} = \langle V, M_{m+1}, p_{m+1} \rangle, \ldots, \pi_{n-1} = \langle V, M_{n-1}, p_{n-1} \rangle$.
- Type 3: $\pi_n = \langle W, X, n_1, \ldots, n_m \rangle$.

$$(\delta_2, \ldots, \delta_n) = (t, \ldots, t, t-1, \ldots, t-1, t+1-m).$$

Thank you very much for your attention!