DARNEC'15

Design and Application of Random Network Codes

Istanbul Technical University

Access-optimal MSR codes with optimal sub-packetization over small fields

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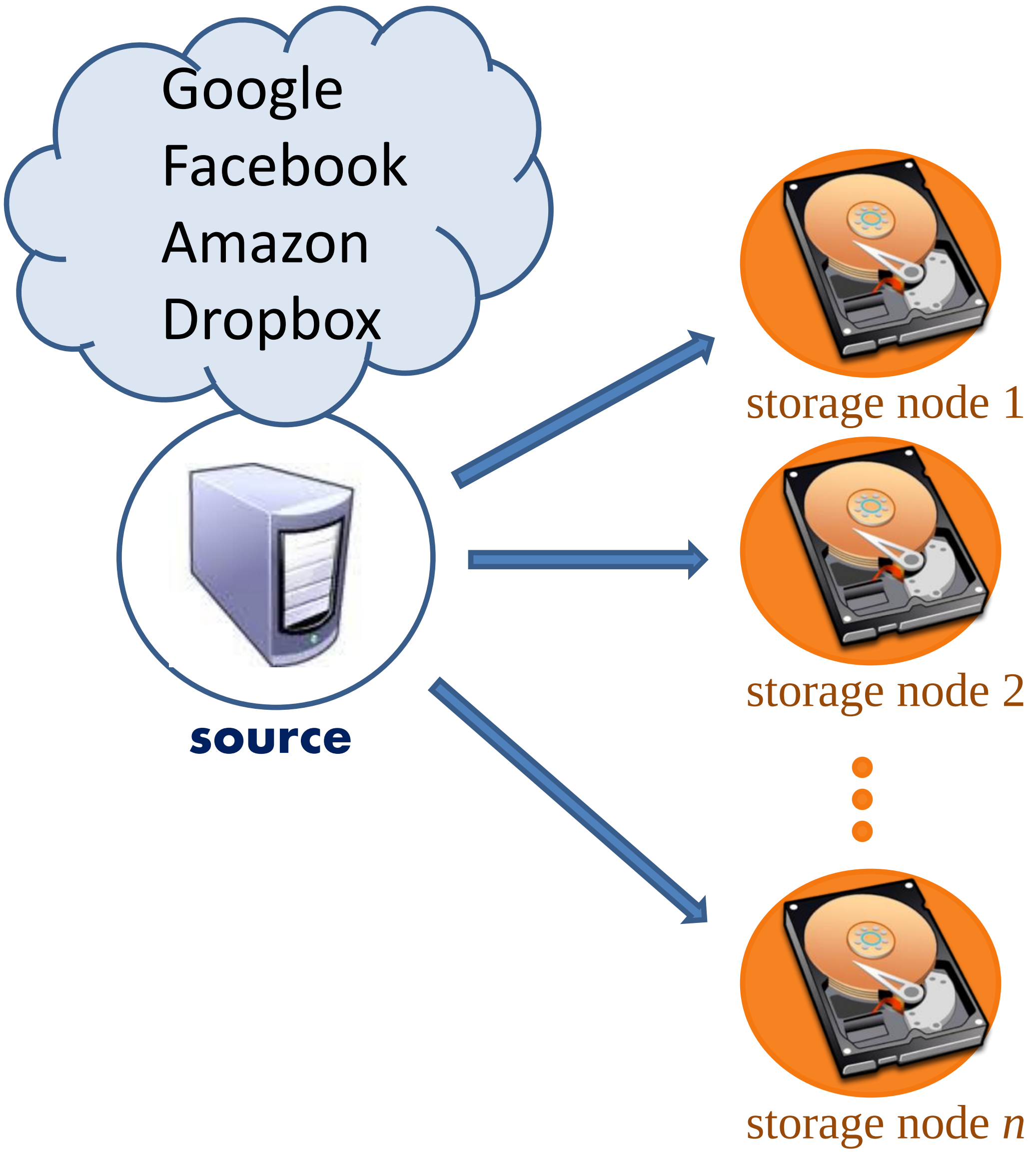
Joint work with

Netanel Raviv and Tuvi Etzion

Technion
Israel Institute of Technology

Distributed storage system (DSS)



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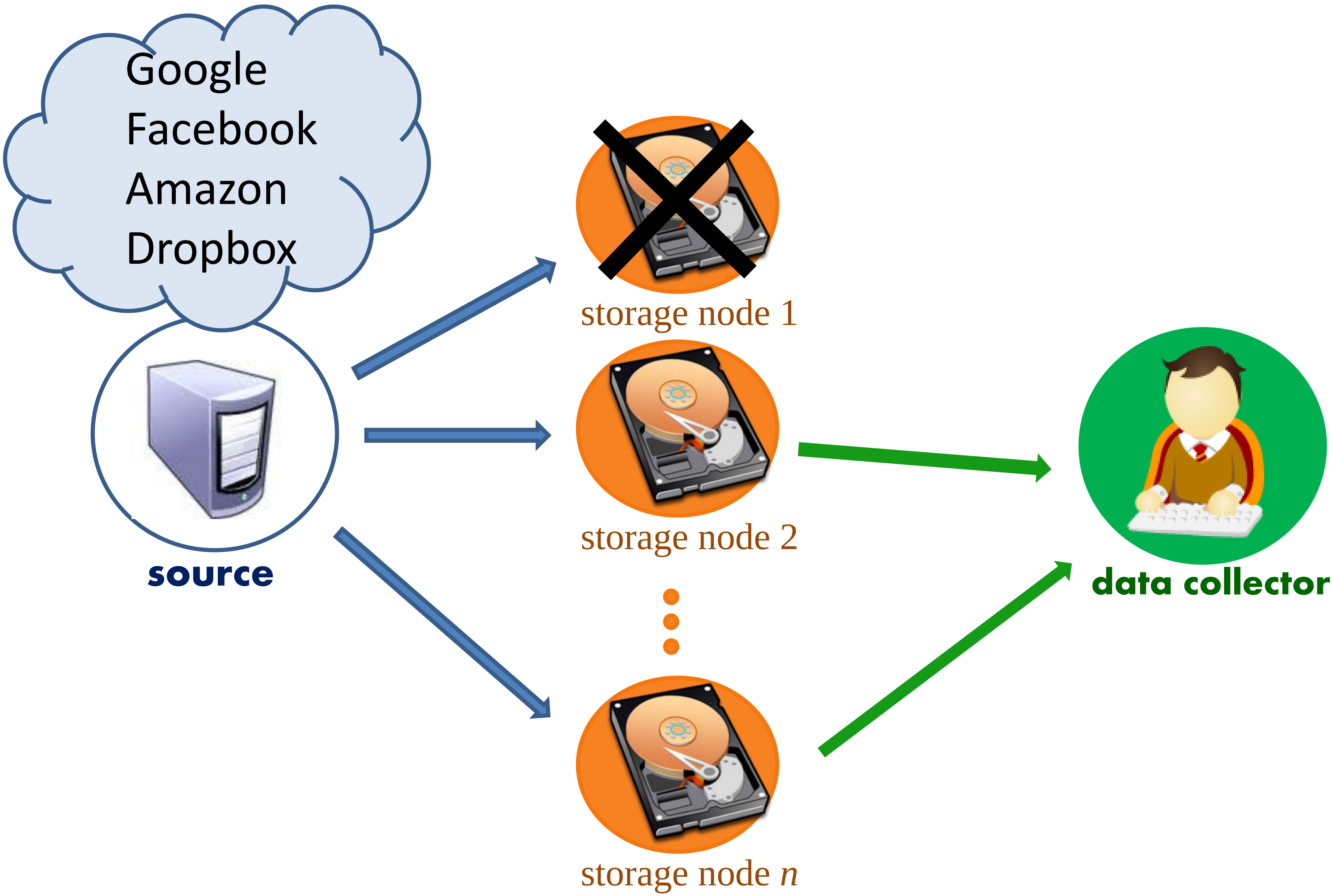
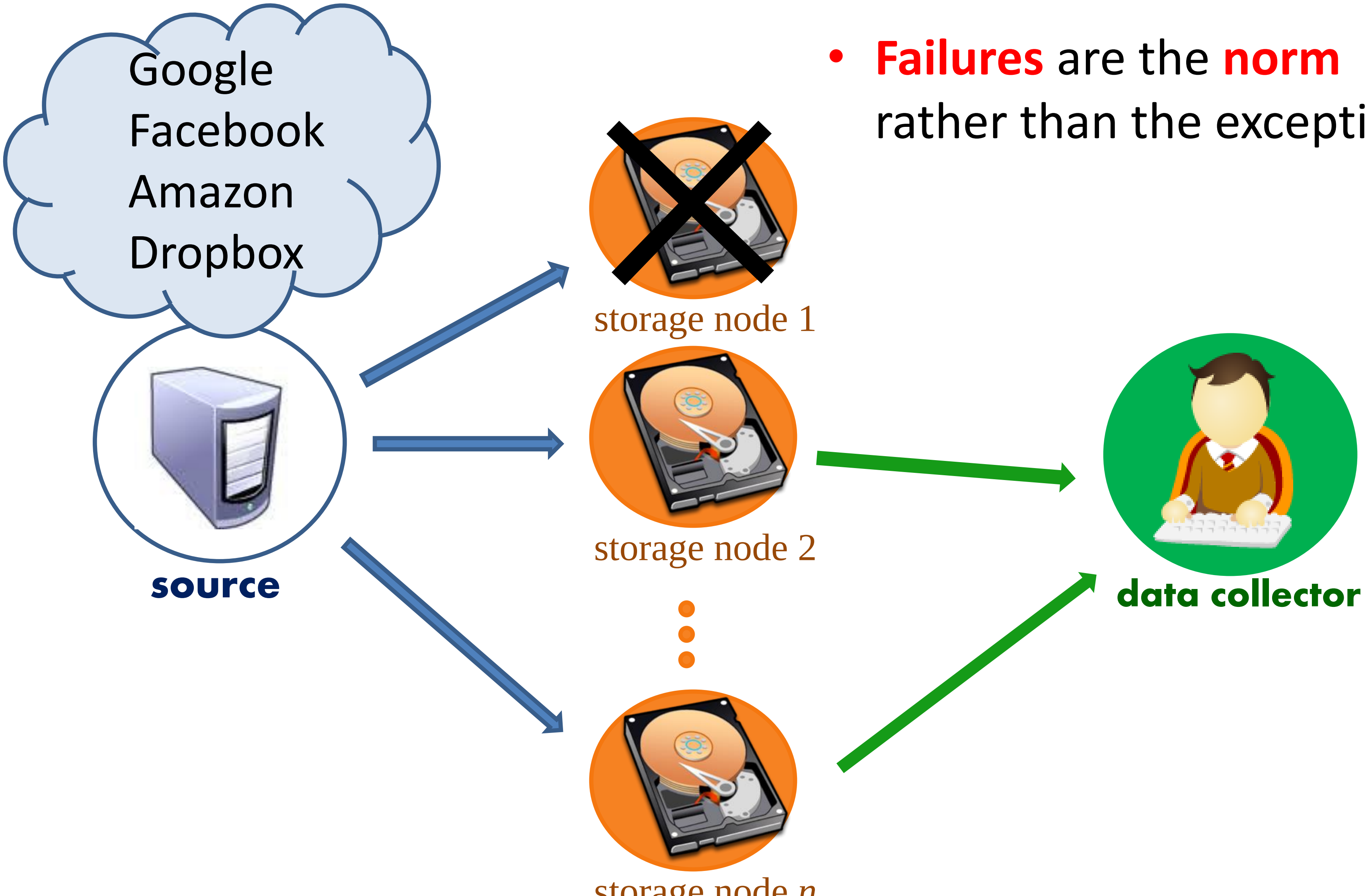


Distributed storage system (DSS)

- **Failures** are the **norm** rather than the exception



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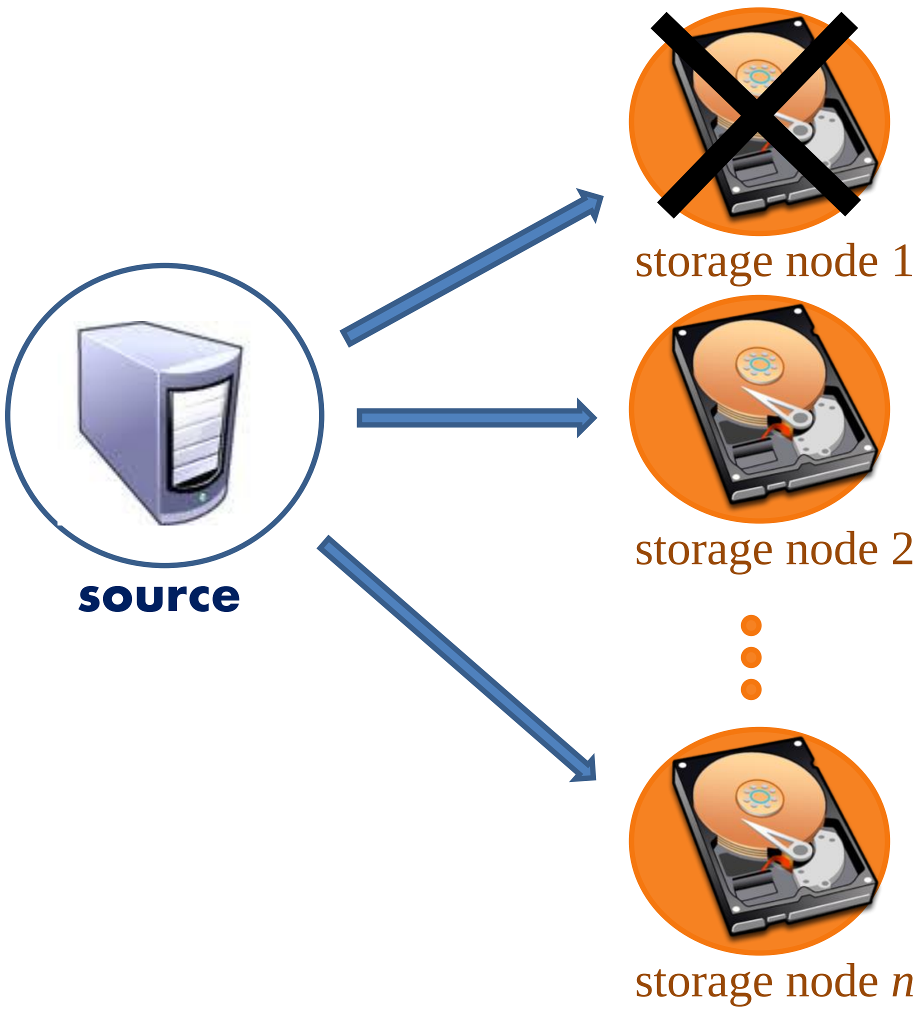
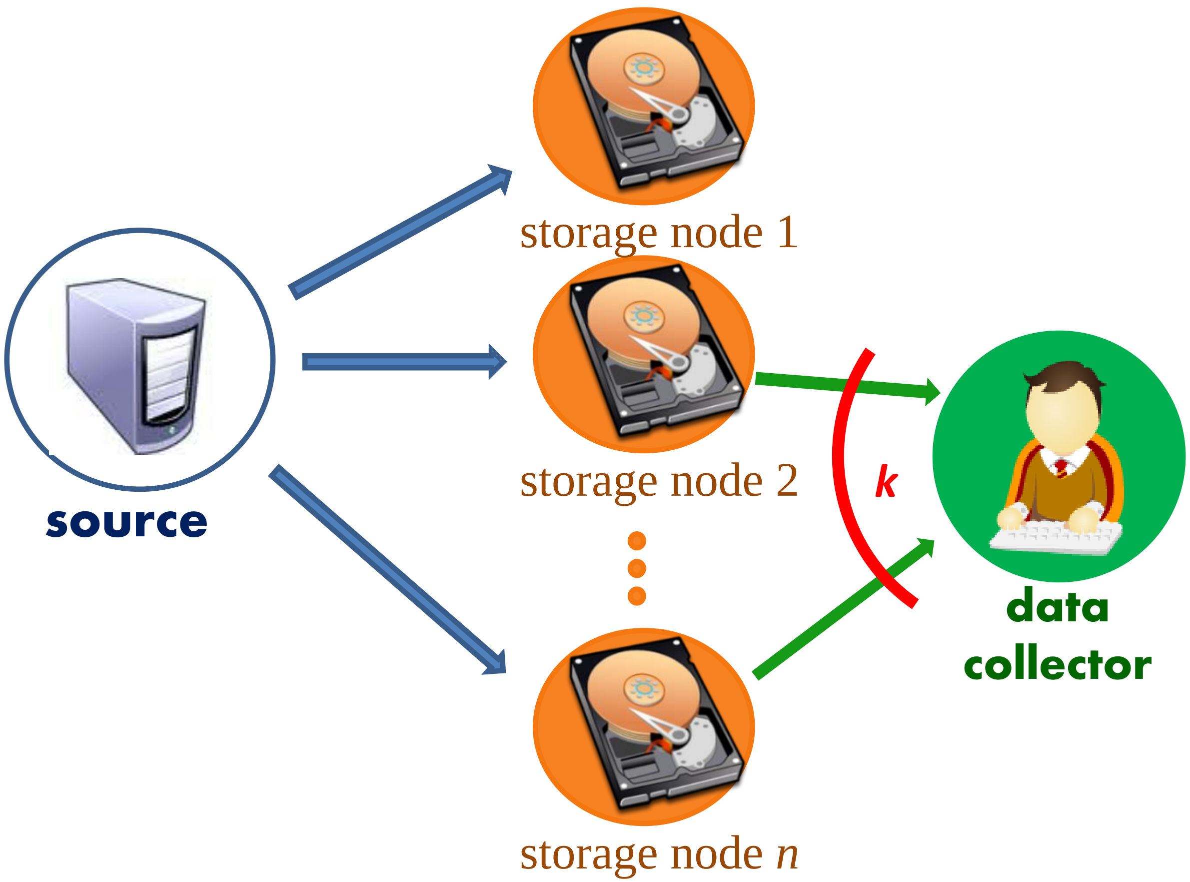


- **Failures** are the **norm** rather than the exception
- **Redundancy** for reliability
  - **Replication**
  - **Erasure coding**

Coding for distributed storage system



Using an (n, k) MDS code:

- Partition the original data into k packets.
- Generate n packets. Store each packet in a different node.

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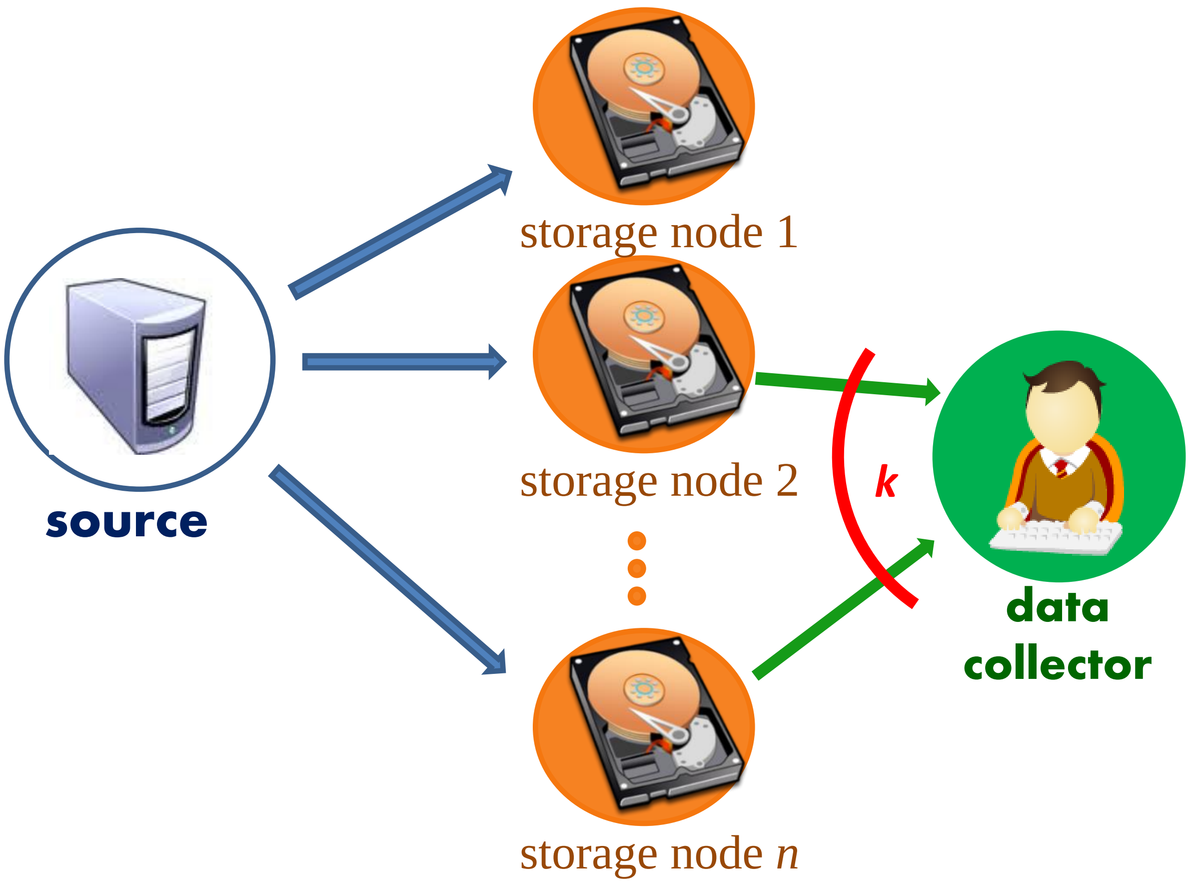

Using an (n, k) MDS code:

- Partition the original data into k packets.
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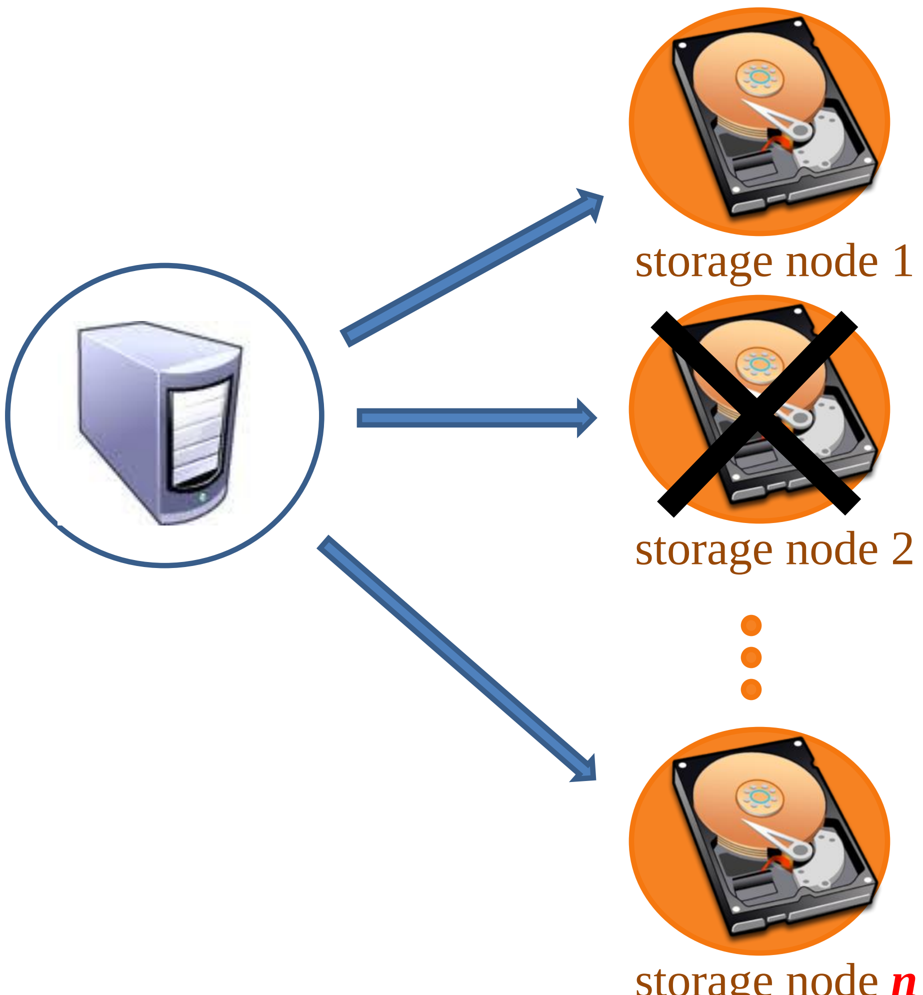
(n, k) **MDS property**: reconstruct the stored data from any k nodes

$\Updownarrow$

The system can tolerate any $n - k$ node failures

Node repair



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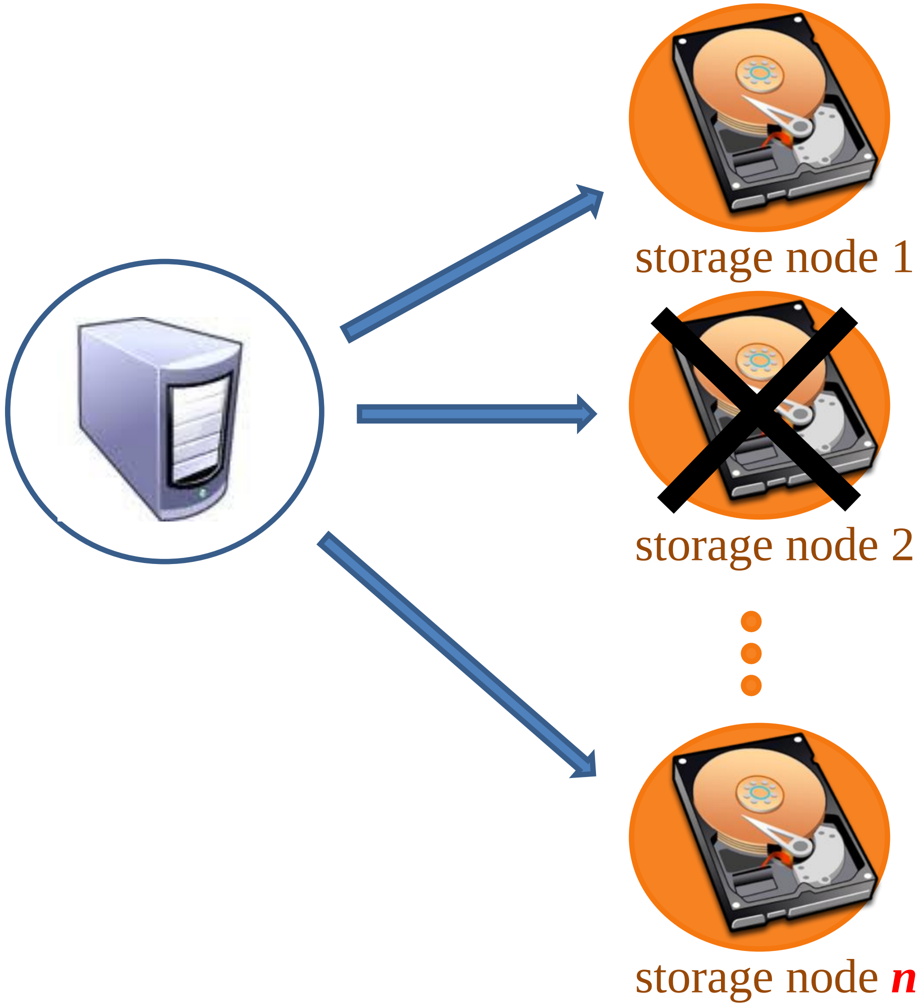
Node repair



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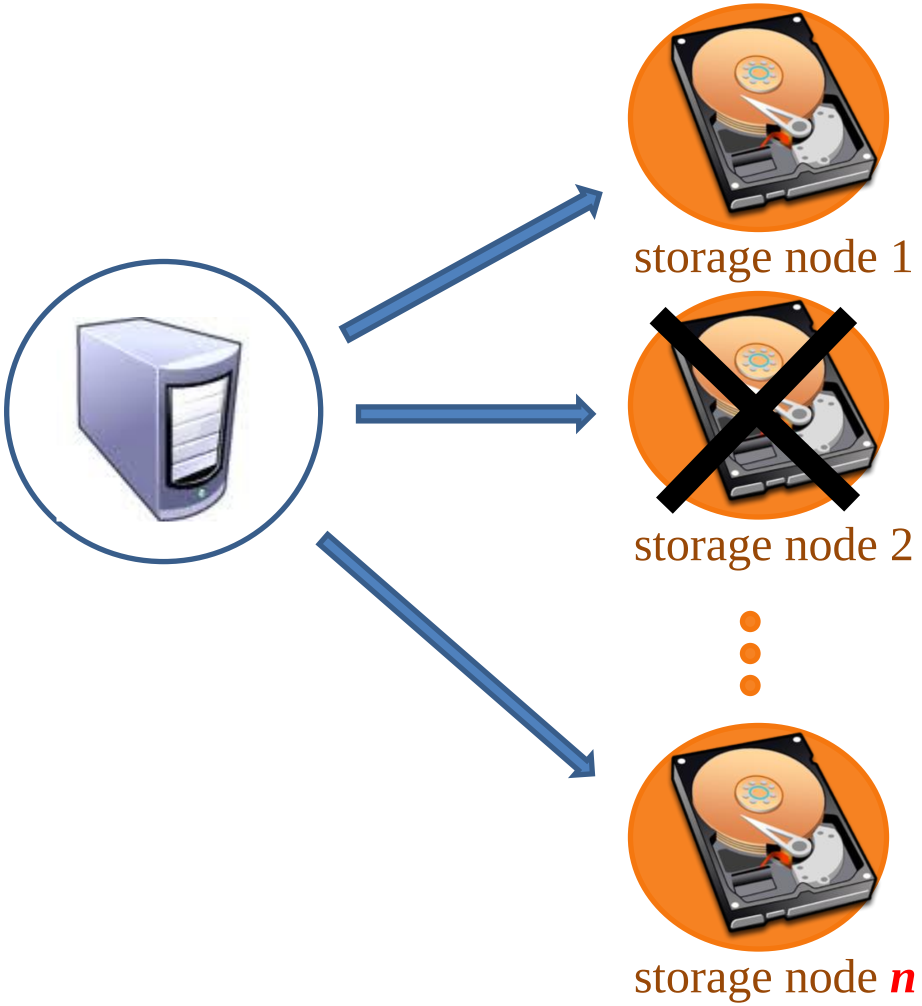

- If only **one** node fails, how to rebuild the redundancy?
- **Naïve method:** to reconstruct the whole data from ***k*** nodes
- <u>Solution:</u> ***regenerating codes***[DimGodWaiRam07] for efficient single node repairs

Regenerating codes – the idea

- To reduce the repair bandwidth, during a node repair
  - communicate with more nodes,
  - but download only part of their stored data

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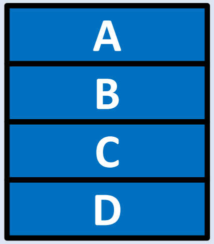

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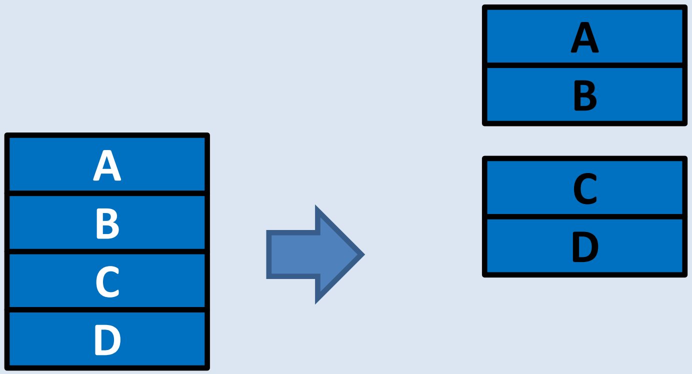

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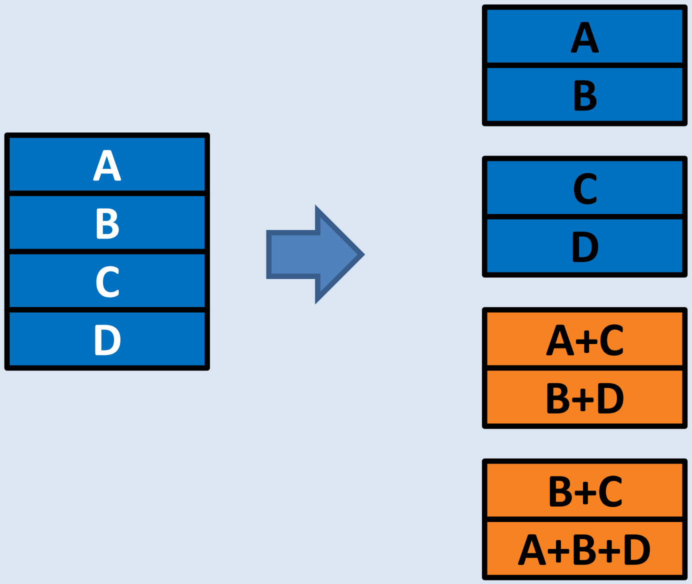

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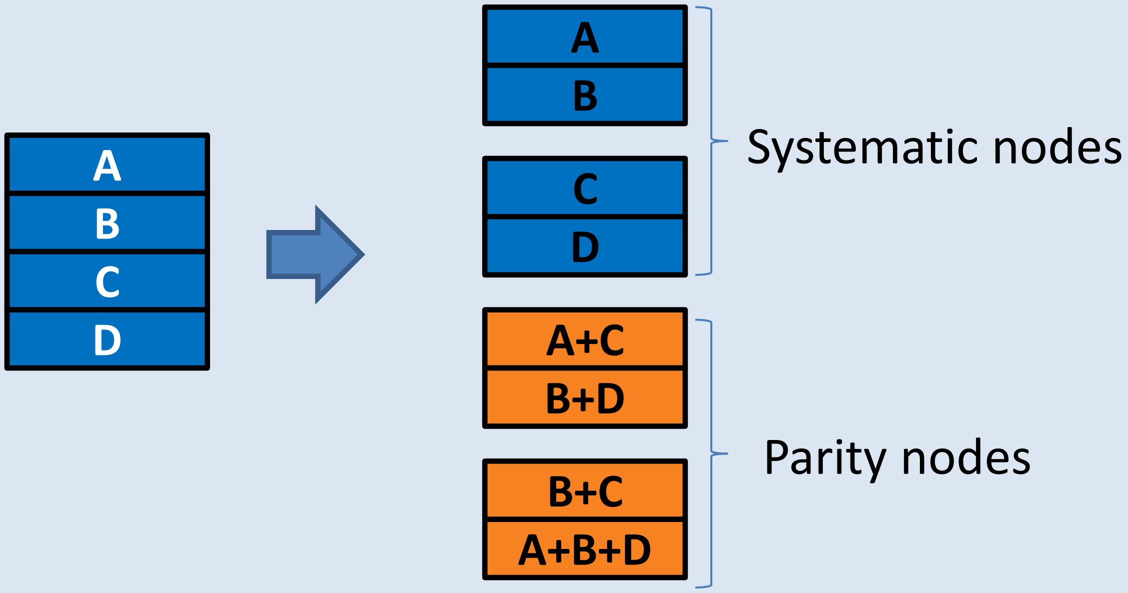

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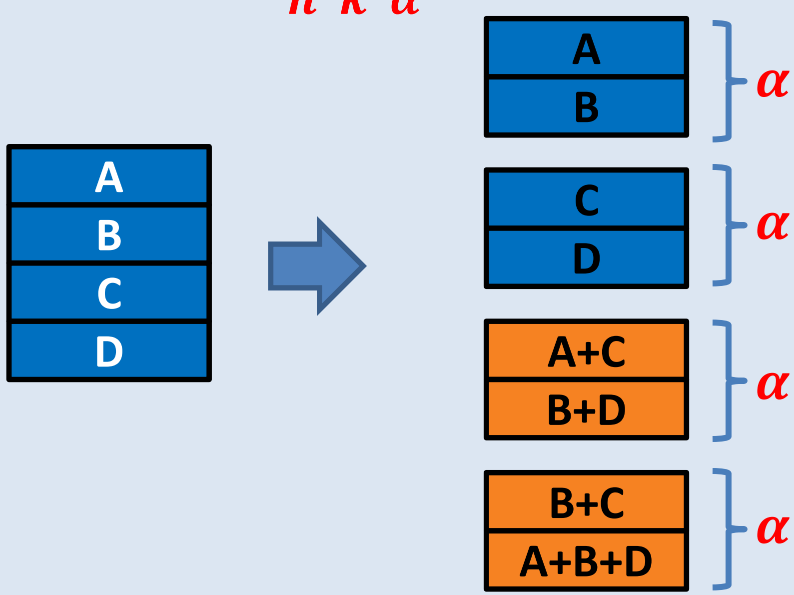
$n\ k\ \alpha$



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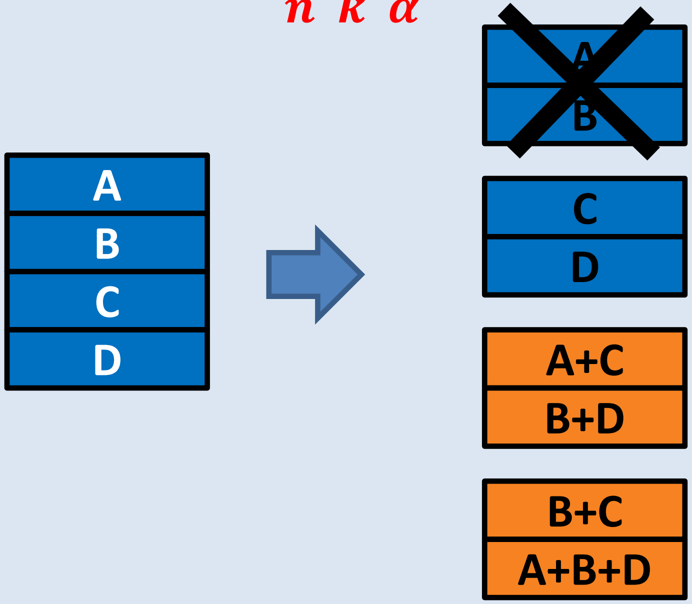

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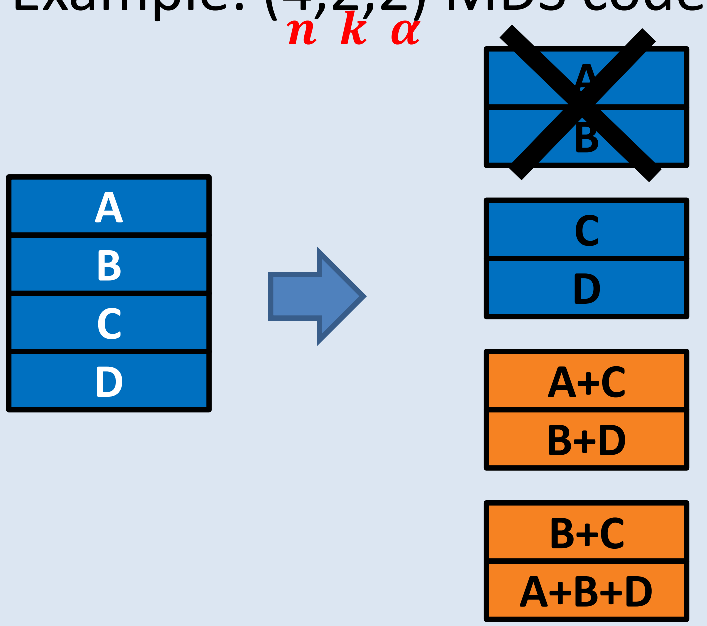

Naïve repair:

read from any k =2 nodes => **4** symbols

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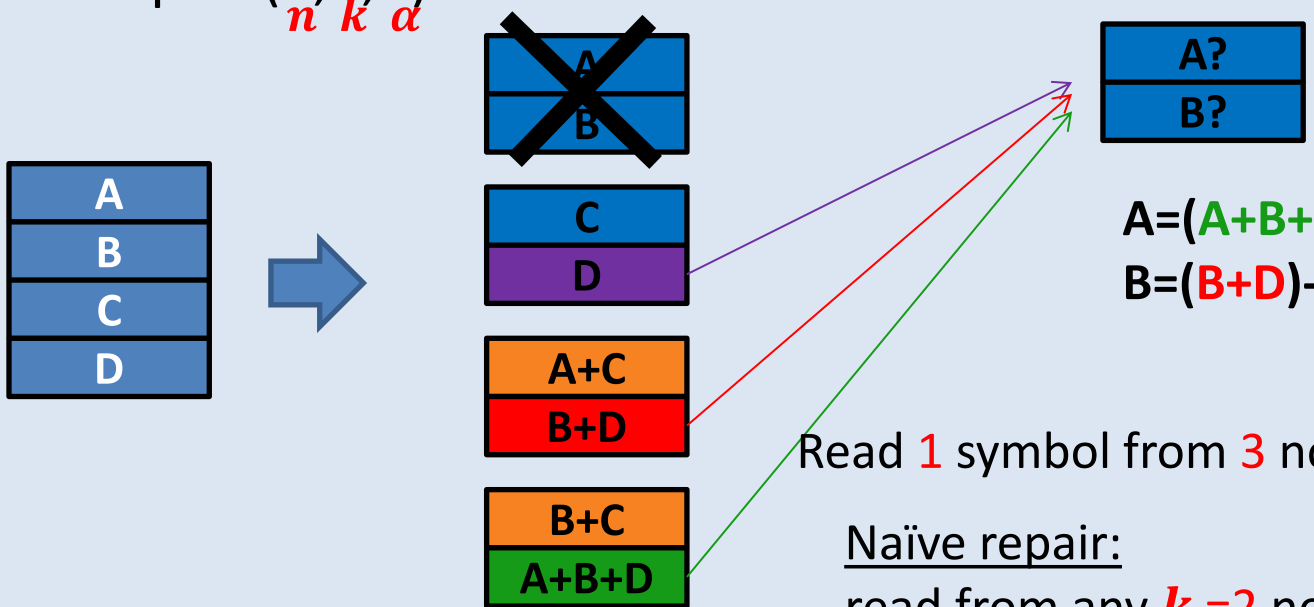


A=(A+B+D)-(B+D)

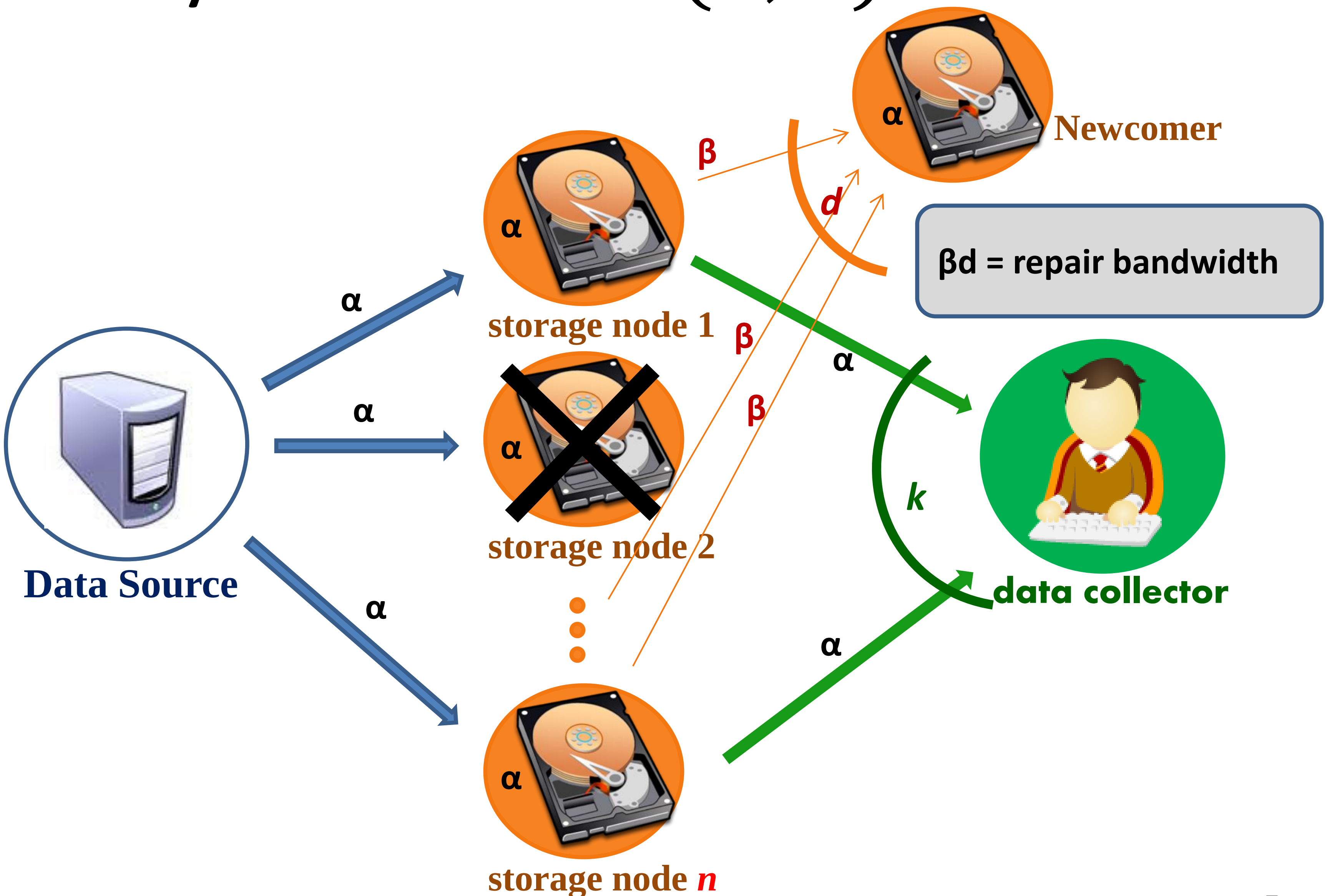
B=(B+D)-D

Read 1 symbol from 3 nodes => 3 symbols

<u>Naïve repair:</u>

read from any k =2 nodes => 4 symbols

System model: (n, k)-DSS



Storage-repair bandwidth tradeoff

- [DimGodWaiRam07] established a tradeoff between storage α vs. repair bandwidth $\gamma = \beta d$ for an (n, k) - DSS
- 2 extreme points of the trade-off:

MSR (minimum storage) : $(\alpha_{MSR}, \gamma_{MSR}) = \left(\frac{M}{k}, \frac{Md}{k(d-k+1)}\right)$

MBR (minimum bandwidth) : $(\alpha_{MBR}, \gamma_{MBR}) = \left(\frac{2Md}{2kd-k^2+k}, \frac{2Md}{2kd-k^2+k}\right)$

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Each node sends a fraction $\frac{1}{n-k}$ of its stored symbols

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MDS array codes

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EXAMPLE: HDFS RAID RS code vs. MSR/MBR

640 MB file => 10 blocks

(14,10) –RS code

1 2 3 4 5 6 7 8 9 10

P1 P2 P3 P4

Data size M = 640 MB

Storage nodes n = 14

DC connects to any k = 10 nodes

Newcomer node connects to d = 10 nodes

	RS code	MSR code	MBR code
Storage per node α	**64 MB**		
Repair bandwidth βd	**640 MB**		

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	RS code	**MSR code**	**MBR code**
Storage per node α	64 MB	**64 MB**	**98 MB**
Repair bandwidth βd	640 MB	**208 MB**	**98 MB**

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6. Minimum sub-packetization factor $\boldsymbol{\alpha = r^{k/r}}$

Optimal access codes

- Accessing the minimum number of helper node's symbols during repairs

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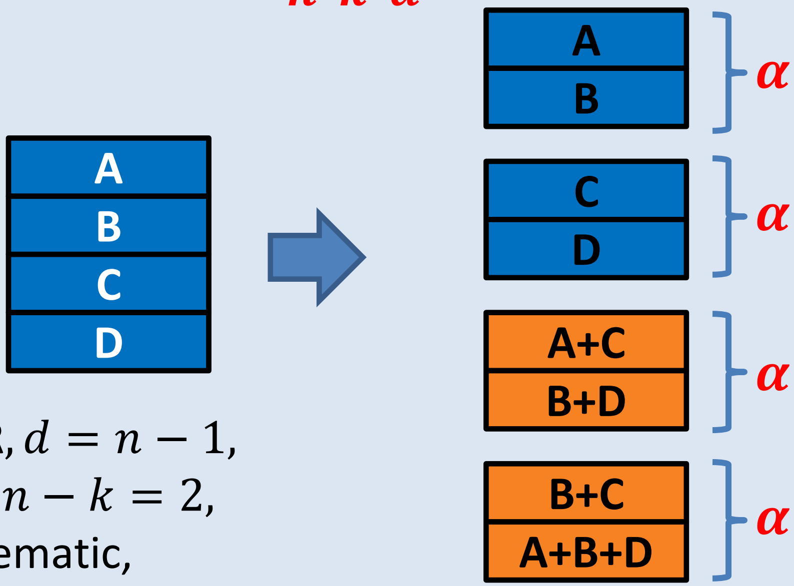
Example: (4,2,2) MDS code
n k α



MSR, $d = n - 1$,
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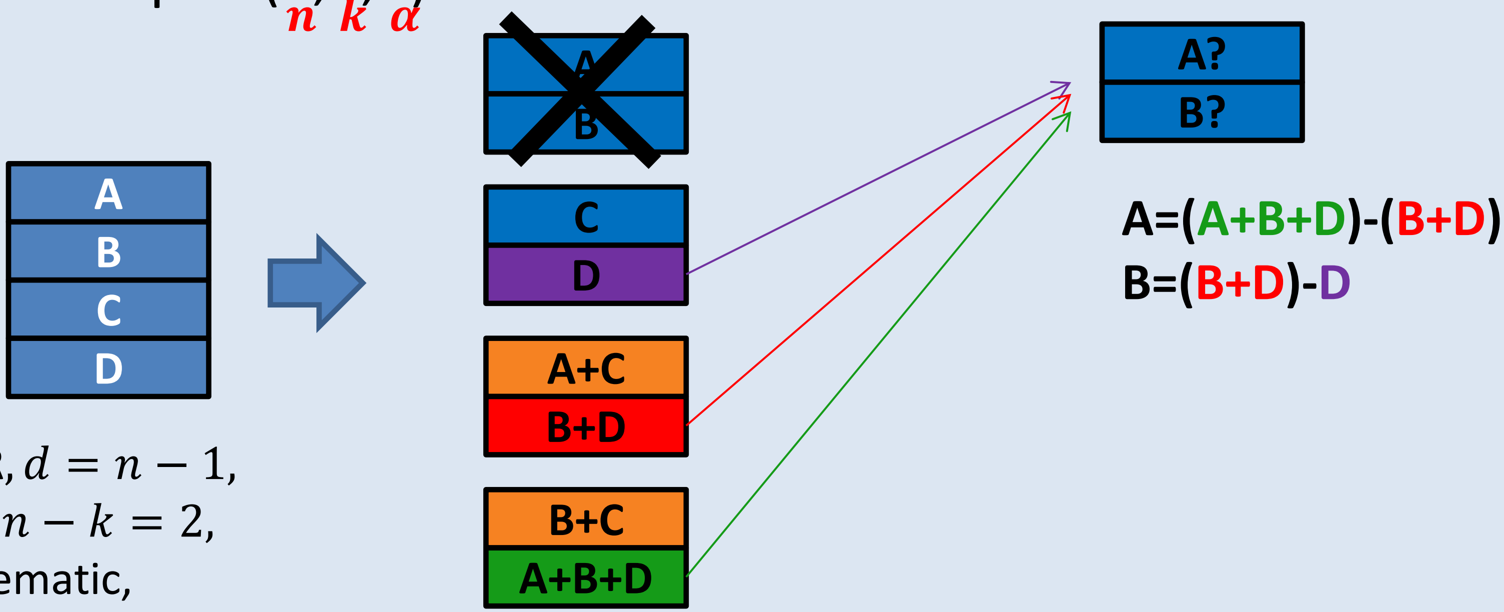

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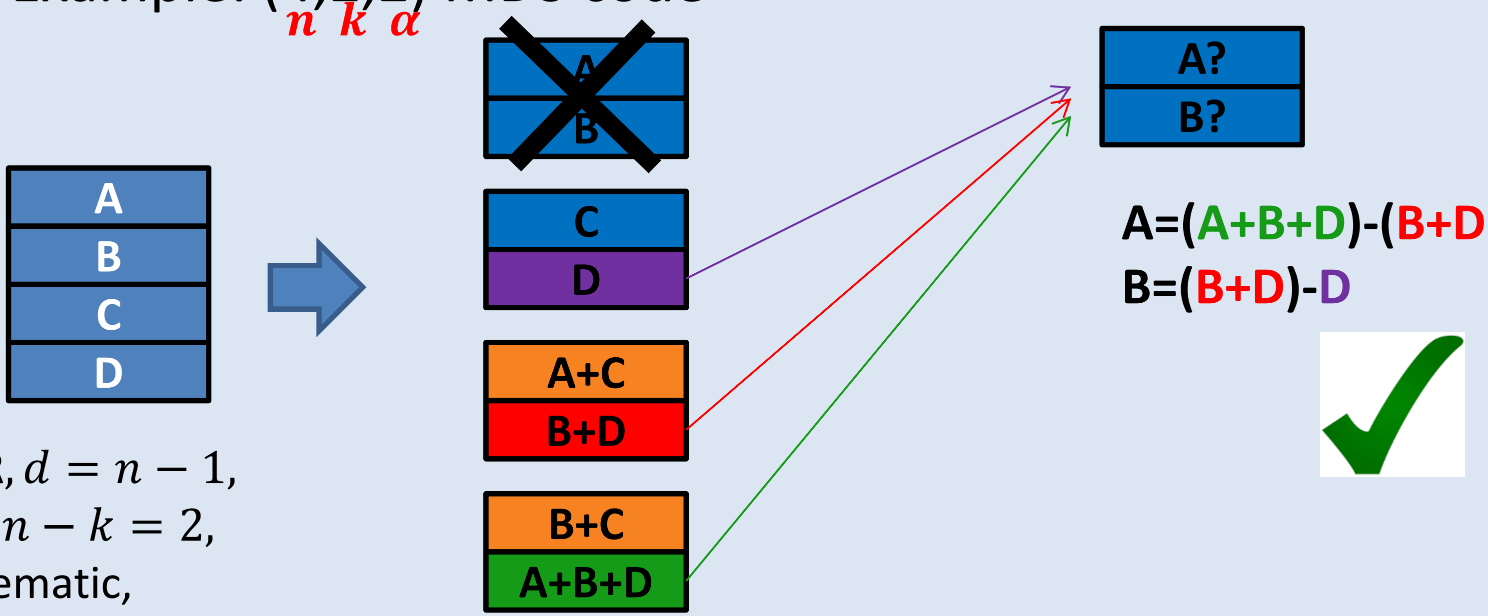

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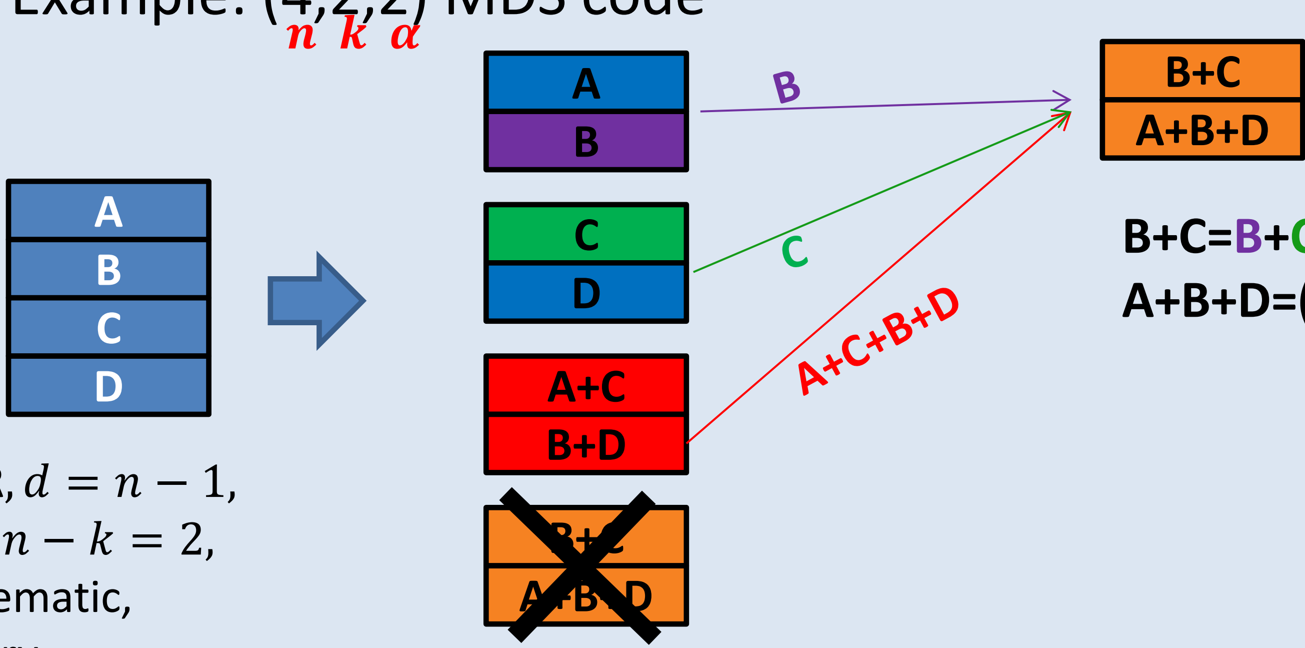

B+C=B+C

A+B+D=(A+C+B+D)+ C

MSR, $d = n - 1$, $r = n - k = 2$, systematic, binary

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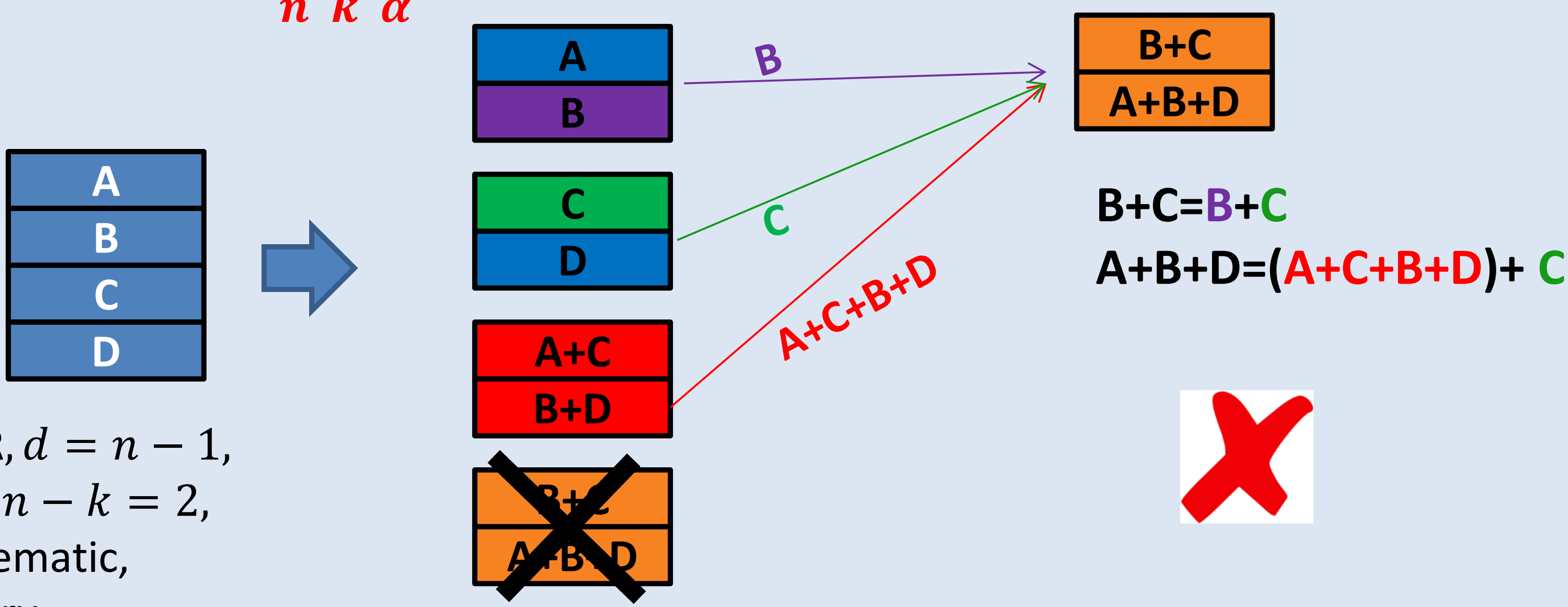

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Our approach

- Subspace condition*
  – The problem of construction of MSR codes is described by the algebraic problem of construction of certain matrices and subspaces
- Graph theory
  – Perfect matchings in complete hypergraphs

* I. Tamo, Z. Wang, and J. Bruck, “Access versus bandwidth in codes for storage", IEEE Trans. Inf. Theory, 2014.

Example: Subspace condition

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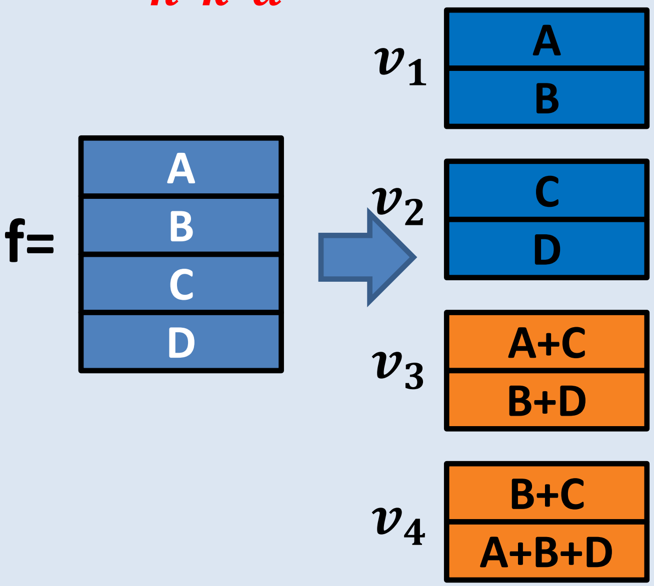

Encoding:

$$v_3 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} v_1 + \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} v_2$$

$$v_4 = \begin{pmatrix} \mathbf{0} & \mathbf{1} \\ \mathbf{1} & \mathbf{1} \end{pmatrix} v_1 + \begin{pmatrix} \mathbf{1} & \mathbf{0} \\ \mathbf{0} & \mathbf{1} \end{pmatrix} v_2$$

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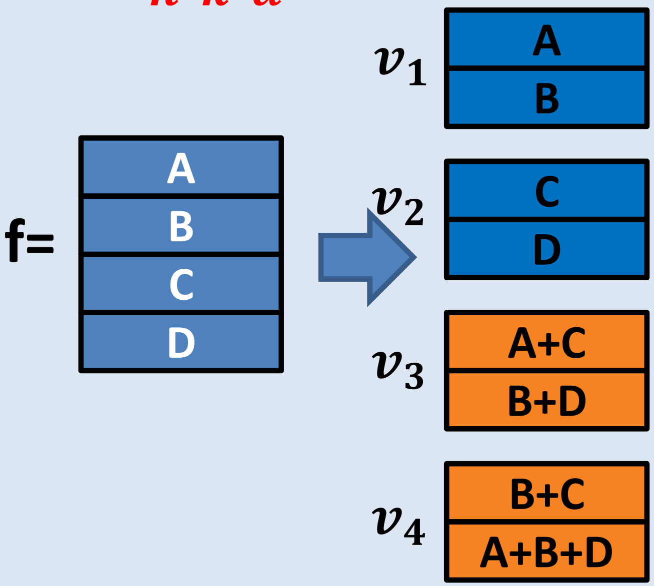

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A_1, A_2, $A_1 - A_2$ **are invertible**

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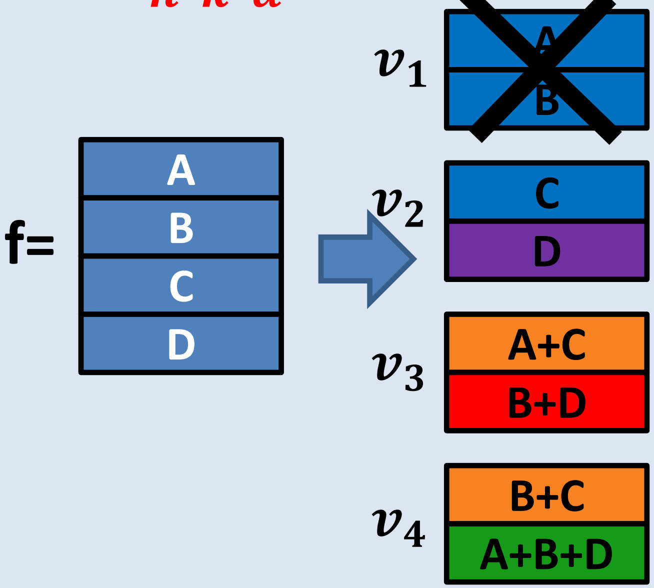

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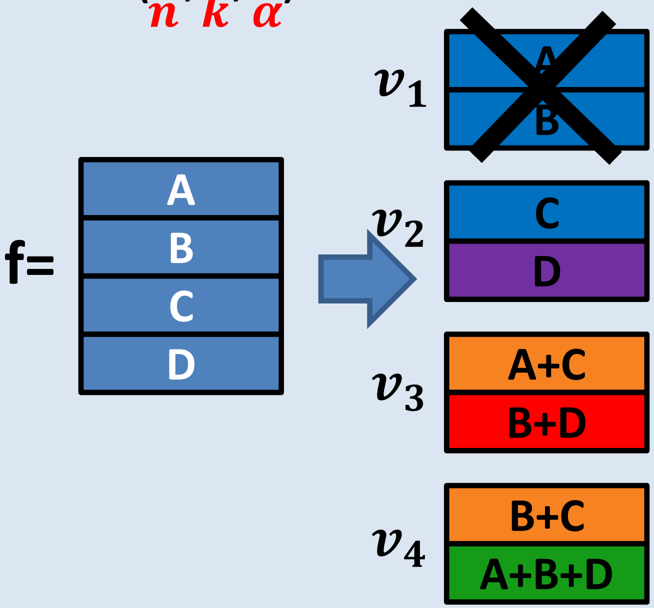

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To repair node 1, node j sends $S_{1j}v_j$ and new node 1 gets

$$\begin{pmatrix} S_{12}v_2 \\ S_{13}v_3 \\ S_{14}v_4 \end{pmatrix} = \begin{pmatrix} S_{12}v_2 \\ S_{13}(v_1+v_2) \\ S_{14}(A_1v_1 + A_2v_2) \end{pmatrix} = \begin{pmatrix} 0 & S_{12} \\ S_{13} & S_{13} \\ S_{14}A_1 & S_{14}A_2 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}$$

$S_{12} = S_{13} = S_{14} = (0,1)$

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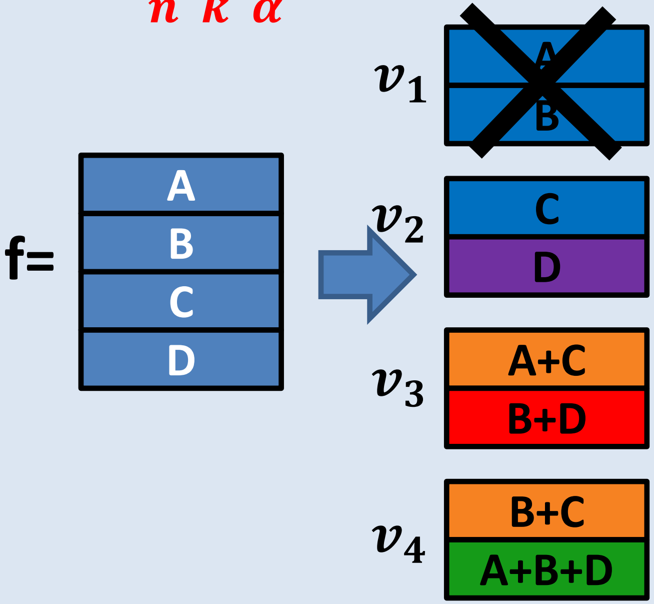

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$$\begin{pmatrix} S_{12}v_2 \\ S_{13}v_3 \\ S_{14}v_4 \end{pmatrix} = \begin{pmatrix} S_{12}v_2 \\ S_{13}(v_1+v_2) \\ S_{14}(A_1v_1 + A_2v_2) \end{pmatrix} = \begin{pmatrix} 0 & S_{12} \\ S_{13} & S_{13} \\ S_{14}A_1 & S_{14}A_2 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}$$

$S_{12} = S_{13} = S_{14} =$(0,1)

To recover v_1 it should hold

1. rank$\begin{pmatrix} S_{13} \\ S_{14}A_1 \end{pmatrix} = 2$
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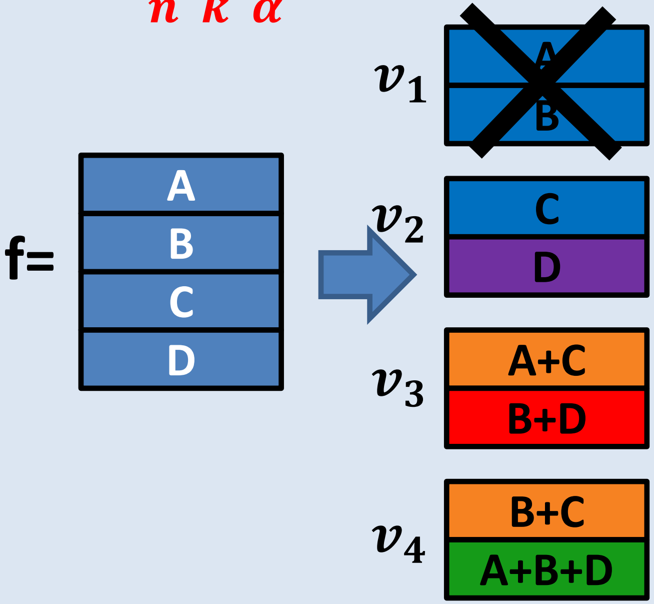
Example: Subspace condition

(4,2,2) MSR code
n k α



Encoding:

$$v_3 = \begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix} v_1 + \begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix} v_2$$

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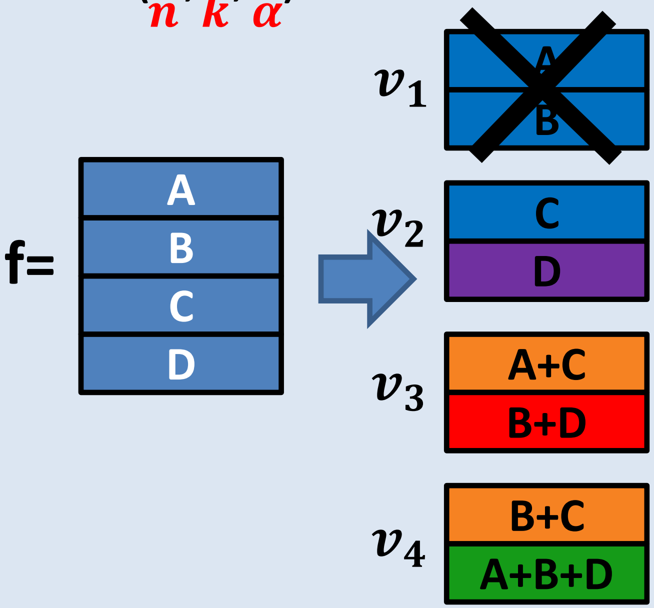

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Subspace condition

Subspace condition for $(n, k, \alpha) - MSR$ code

- **Encoding:** a file $f \in \mathbb{F}_q^{k\alpha}$ is partitioned into k parts of size α:

$$\mathbf{f} = (v_1, v_2, \dots, v_k), v_i \in \mathbb{F}_q^{\alpha}$$

$$\mathbf{f} \to \mathbf{f} \cdot G \quad = (C_1, \dots, C_n) \quad ,$$

where $G = \begin{pmatrix} \mathrm{I} & & & \mathrm{I} & A_{11} \cdots A_{(n-k-1)1} \\ & \ddots & & \vdots & \vdots \quad \ddots \quad \vdots \\ & & \mathrm{I} & \mathrm{I} & A_{1k} \cdots A_{(n-k-1)k} \end{pmatrix}$, $A_{ij} \in \mathbb{F}_q^{\alpha \times \alpha}$

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- **Repair of node t:**

  node $j \in [n] \backslash \{\text{t}\}$ sends $S_t\, C_j$, where $S_t \in \mathbb{F}_q^{\frac{\alpha}{r} \times \alpha}$, $r = n - k$.

  $\mathbb{S}_t := < S_t >$ is called the repairing subspace

  (the repairing subspace are independent on the helper node*)

  $\mathbb{S}_t \subseteq \mathbb{F}_q^{\alpha}$, $\dim(\mathbb{S}_t) = \frac{\alpha}{r}$.

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1. $\mathbb{S}_t + \mathbb{S}_t A_{1t} + \ldots + \mathbb{S}_t A_{(r-1)t} = \mathbb{F}_q^{\alpha}$

The independence property

2. $\mathbb{S}_t = \mathbb{S}_t A_{ij}, j \neq t,$ where $\mathbb{S}_t A_{it} := \mathrm{rs}(S_t A_{it})$

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Sufficient Condition

- **Theorem** [TWB14]: Let α and r be integers s.t. $r|\alpha$. If there exist subspaces $\mathbb{S}_1, \dots, \mathbb{S}_k \subseteq \mathbb{F}_q^{\alpha}$ of dimension α/r and encoding matrices $A_{ij} \in \mathbb{F}_q^{\alpha\times\alpha}, i \in [r-1], j \in [k]$ which satisfy

1. The nonsingular property:

   Every square block submatrix of $\begin{pmatrix} I & A_{11} & \cdots & A_{(r-1)1} \\ \vdots & \vdots & \ddots & \vdots \\ I & A_{1k} & \cdots & A_{(r-1)k} \end{pmatrix}$ is invertible

2. The independence property:

$$\mathbb{S}_t + \mathbb{S}_t A_{1t} + \dots + \mathbb{S}_t A_{(r-1)t} = \mathbb{F}_q^{\alpha}$$

3. The invariance property:

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Then the corresponding code is an (n, k, α) −MSR code.

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- For a given node capacity of α symbols and a given number of parity nodes r, what is the maximum number of systematic nodes k for an optimal access MSR code?
- **Theorem** [TWB14]**:** Let k be the largest number of systematic nodes in an $(k+r, k, \alpha)$ optimal access MSR code, then

$$k = r\log_r \alpha$$

Our goal

For $k = r\log_r \alpha,$ construct sets of

- subspaces $\mathbb{S}_1, \ldots, \mathbb{S}_k \subseteq \mathbb{F}_q^{\alpha}$ of dimension α/r spanned by vectors of the standard basis
- Invertible matrices $A_{ij} \in \mathbb{F}_q^{\alpha\times\alpha}, 1 \leq i \leq r-1, 1 \leq j \leq k$

which satisfy

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for a field size q as small as possible.

In particular for $A_{ij} = (A_j)^i$:

For $k = r\log_r \alpha$, construct a set $\{(\boldsymbol{A_i}, \mathbb{S}_i)\}_{\mathbf{i=1}}^{\mathbf{k}}$ of

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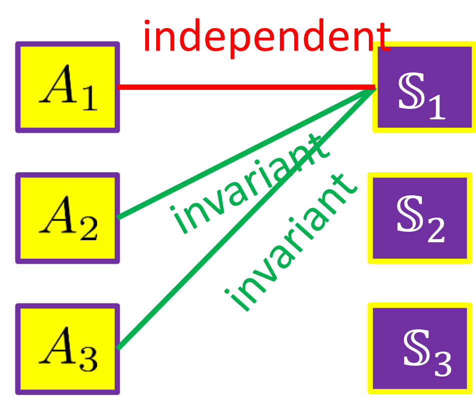

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Known results

Constructions of access-optimal MSR codes with the minimum sub-packetization factor $\boldsymbol{\alpha = r^{k/r}}$:

number r of parity nodes	finite field size [TWB 12]	finite field size [ASK15]
2	$k+1$	$k+1$
3	$k^2 3^{k/3-1}+1$	$\binom{k+3}{k} 3^{k/3+1}$
r	$k^{r-1} r^{k/r-1}+1$	$\binom{n}{k} r^{k/r+1}$

[TWB 12] I. Tamo, Z. Wang, and J. Bruck, "Long MDS codes for optimal repair bandwidth," ISIT12.
[ASK15] G. K. Agarwal, B. Sasidharan, and P. V. Kumar, "An alternative construction of an access-optimal regenerating codes with optimal sub-packetization level," NCC 2015.

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Our Construction

- Consider the matrix

$$A \triangleq \begin{pmatrix} 0 & 0 & \cdots & 1 \\ 1 & 0 & \cdots & 0 \\ & & & \\ 0 & \ddots & 0 & 0 \\ 0 & \cdots & 1 & 0 \end{pmatrix} \in \mathbb{F}_q^{r \times r}$$

$$\mathrm{minpoly} A = x^r - 1$$

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- The subspace $\langle e_0 \rangle$ is an ***independent*** subspace for *A*.

The independence property:

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  - The eigenvalues of *A* are the roots of unity of order r

    $$\gamma_0 = 1, \gamma_1, \ldots, \gamma_{r-1}$$

  - The eigenvectors are

    $$\left\{ \left(1, \gamma_i, \gamma_i^2, \cdots, \gamma_i^{r-1}\right) \right\}_{i=0}^{r-1}.$$

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- Let $\mathrm{diag}_{\alpha/r}(A) = \begin{pmatrix} A & & \\ & \ddots & \\ & & A \end{pmatrix}$ $\quad A \triangleq \begin{pmatrix} 0 & 0 & \cdots & 1 \\ 1 & 0 & \cdots & 0 \\ 0 & \ddots & 0 & 0 \\ 0 & \cdots & 1 & 0 \end{pmatrix}$
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- Let $M \triangleq P_M^{-1} \mathrm{diag}_{\alpha/r}(A) P_M = \begin{pmatrix} p_0 \\ \vdots \\ p_{\alpha\text{-}1} \end{pmatrix}^{-1} \begin{pmatrix} A & & \\ & \ddots & \\ & & A \end{pmatrix} \begin{pmatrix} p_0 \\ \vdots \\ p_{\alpha\text{-}1} \end{pmatrix}$
- The subspace $\langle \{p_0, p_r, p_{2r}, \ldots\} \rangle$ is an ***independent*** subspace for *M*.
- Subspaces of the form $\left\langle \{p_{ri} + \gamma_j p_{ri+1} + \gamma_j^2 p_{ri+2} + \ldots\}_{i=0}^{\alpha/r-1} \right\rangle$ are eigenspaces, and thus also ***invariant*** subspaces.

Problems:

- Choose the change-of-basis matrices P_M such that the ***invariance*** and the ***independence*** property (for all M's) are satisfied.
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Solution:

Based on perfect matchings in uniform hypergraphs

Perfect matchings in uniform hypergraphs

- r-uniform hypergraph - edges are sets of r vertices
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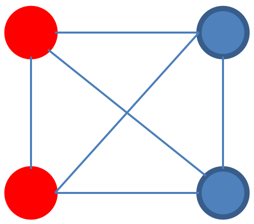

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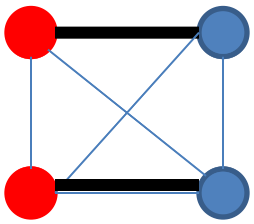

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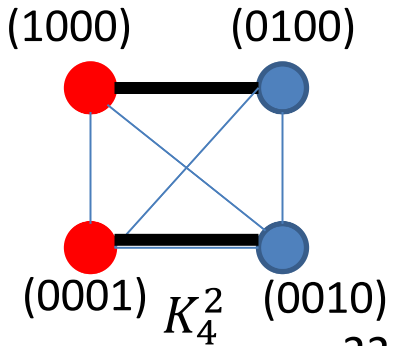
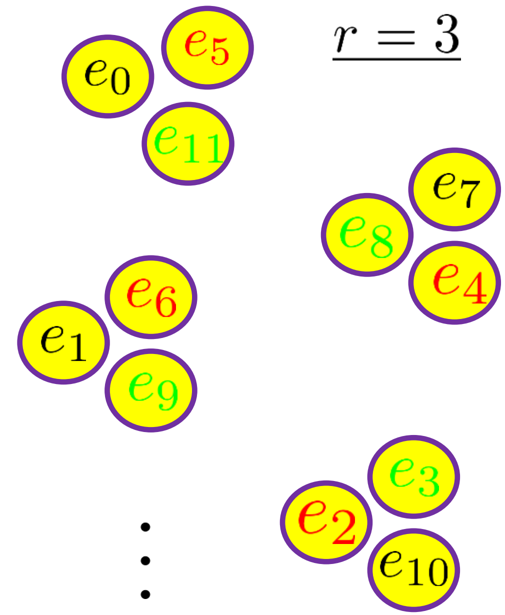
  Perfect colored matching in K_α^r – a perfect matching where no edge contains two vertices of the same color.

- We identify α unit vectors $e_0, \ldots, e_{\alpha-1}$ of length α with the α vertices of K_α^r.



Change Basis Matrices from Matchings

$e_0, \ldots, e_{\alpha-1} \Leftrightarrow \alpha$ vertices of K_α^r



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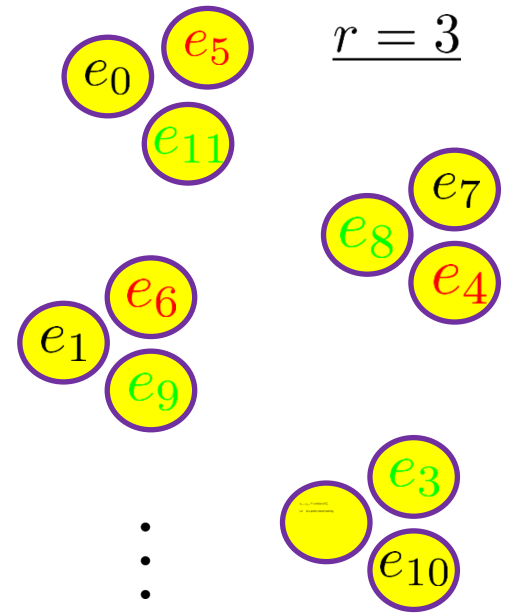

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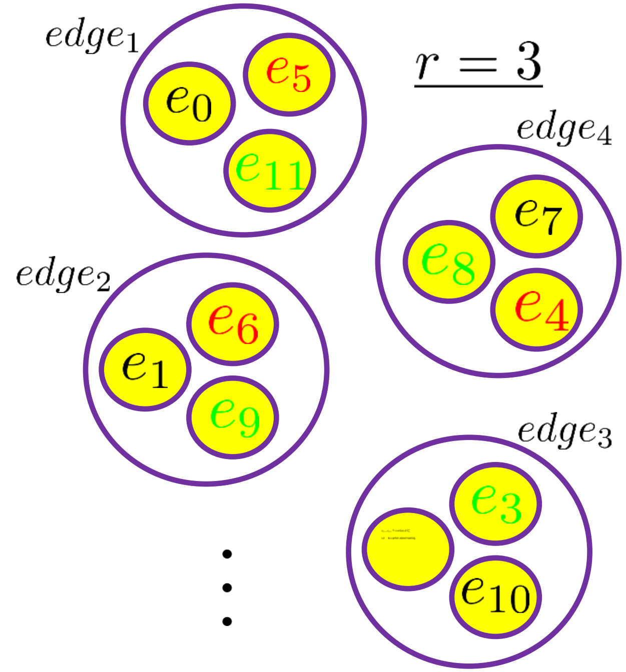

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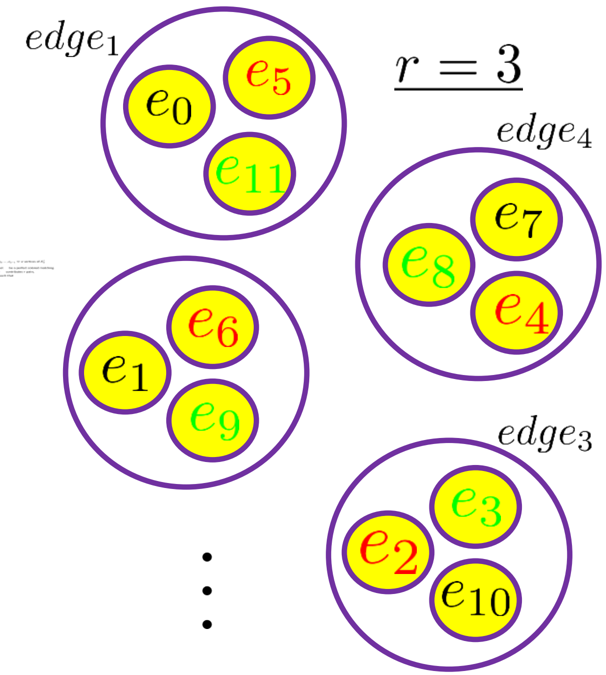

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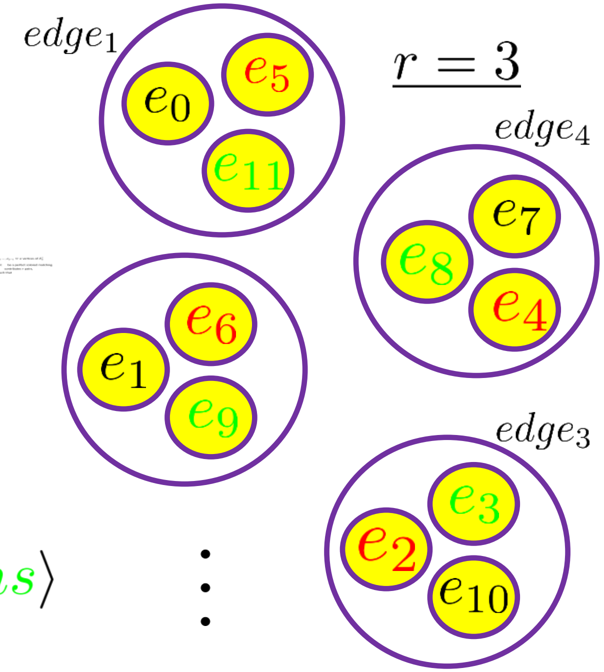

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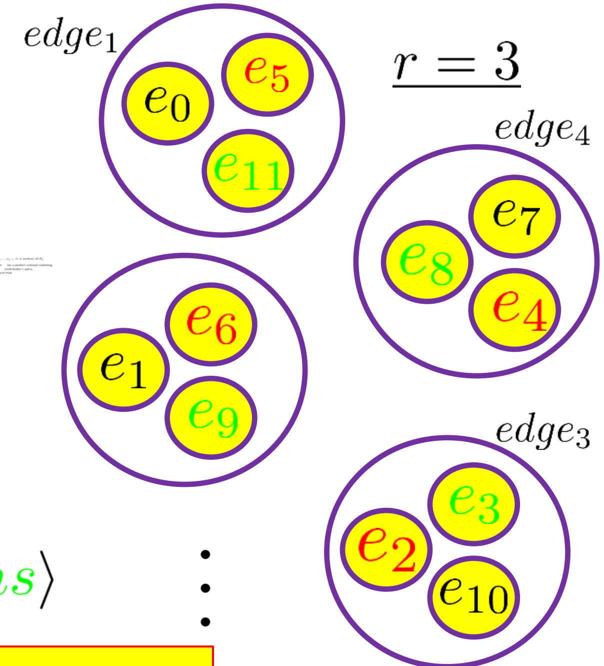

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All repair subspaces are spanned by *unit* vectors then **access-optimal** code

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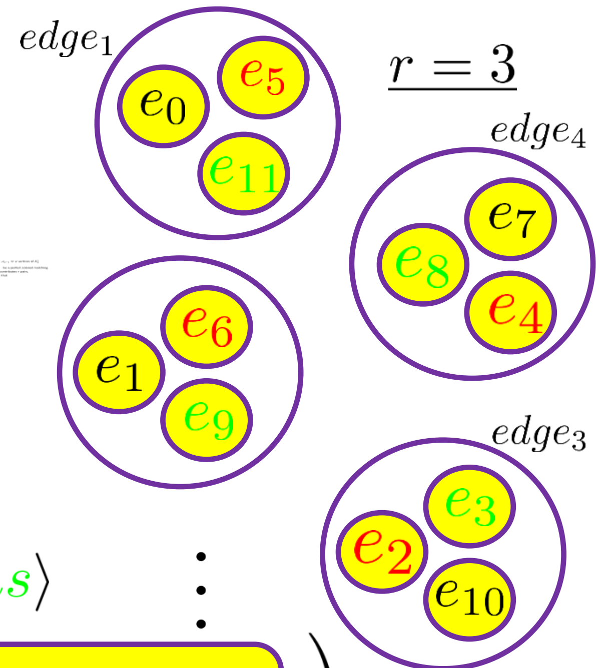

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$$A_{M_i} \triangleq P_{M_i}^{-1} \mathrm{diag}_{\alpha/r}(A) P_{M_i}, \qquad P_{M_i} \triangleq \begin{pmatrix} \boxed{\text{a function of } edge_1} \\ \vdots \\ \boxed{\text{a function of } edge_{\alpha/r}} \end{pmatrix}$$

Change Basis Matrices from Matchings

Ensure that
S_{M_i} is an independent subspace
S_{M_j} is an eigenspace, for $i \neq j$

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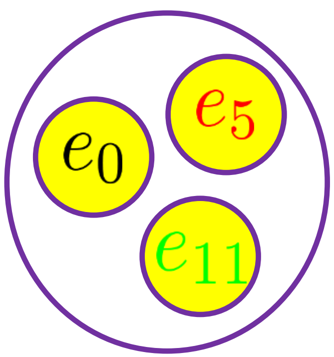

$edge_1$

$$\text{a function of } edge_1 = \begin{pmatrix} 1 & 0 & 0 \\ 1 & -\frac{\gamma_1}{\gamma_1 - 1} & \frac{1}{\gamma_1 - 1} \\ 1 & \frac{1}{\gamma_1 - 1} & -\frac{\gamma_1}{\gamma_1 - 1} \end{pmatrix} \cdot \begin{pmatrix} e_0 \\ e_5 \\ e_{11} \end{pmatrix}$$

For A_{M_2} : S_{M_2} Independent subspace;

S_{M_1} Eigenspace for γ_1

S_{M_3} Eigenspace for γ_2

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The Nonsingular Property

- So far, the construction works for *any* number r of parities.
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  - $\text{rank}(A_i^2 - A_j^2) = \alpha$
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The Nonsingular Property

- <u>Our solution:</u>

  $A_i \Rightarrow \lambda_i A_i \quad, 0 \neq \lambda_i \in \mathbb{F}_q, \lambda_i = \lambda_j$ for the same matching

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- For $\boldsymbol{r = 2}$ parities:

  $$\boldsymbol{q = k/2 + 1}$$

- For $\boldsymbol{r = 3}$ parities:

  $$\boldsymbol{q \geq \begin{cases} 2k + 1 \text{ for odd } q \\ k + 1 \text{ for even } q \end{cases}}$$

Our results

Constructions of access-optimal MSR codes with the minimum sub-packetization factor $\boldsymbol{\alpha = r^{k/r}}$:

number r of parity nodes	finite field size [TWB 12]	finite field size [ASK15]	**Our finite field size q**
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Future Research

- Nonsingularity for more than three parities.
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- <u>In general</u>:

  Non-systematic node failure.

  More than one simultaneous failures?

Thank you!